Magnetostatics

Junqi Zhang

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Conventions

We use SI units. Metric signature is $(+,-,-,-)$. Main reference is [2].

1 Ampere's law and vector potential

1.1 Basic equations from the Biot-Savart law

The Biot-Savart law for magnetic induction made by a single moving charge is

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{dq\mathbf{v} \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{1.1}$$

so for a charge distribution $\rho(\mathbf{r}')$ having velocity $\mathbf{v}(\mathbf{r}')$ the magnetic induction is a integral over source charges

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{1.2}$$

where $\mathbf{J}(\mathbf{r}') \equiv \rho(\mathbf{r}')\mathbf{v}(\mathbf{r}')$ is the current density. With

$$\nabla \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) = -\frac{\mathbf{r}-\mathbf{r}'}{(\mathbf{r}-\mathbf{r}')^3} \tag{1.3}$$

we have

$$\begin{aligned}
\mathbf{B} = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} &= -\frac{\mu_0}{4\pi} \int d^3r' \mathbf{J}(\mathbf{r}') \times \nabla \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) &\quad (1.4)\\
&= -\frac{\mu_0}{4\pi} \int d^3r' \epsilon_{ijk} J_j\left(\mathbf{r}'\right) \partial_k \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) &\quad (1.5)\\
&= -\frac{\mu_0}{4\pi} \int d^3r' \epsilon_{ijk} \partial_k \left(\frac{J_j\left(\mathbf{r}'\right)}{|\mathbf{r}-\mathbf{r}'|} \right) + \frac{\mu_0}{4\pi} \int d^3r' \frac{\epsilon_{ijk}\partial_k J_j\left(\mathbf{r}'\right)}{|\mathbf{r}-\mathbf{r}'|} &\quad (1.6)\\
&= \nabla \times \left(\frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}\left(\mathbf{r}'\right)}{|\mathbf{r}-\mathbf{r}'|} \right) &\quad (1.7)
\end{aligned}$$

where we have used $\nabla \times \mathbf{J}\left(\mathbf{r}'\right) = 0$ in the second last line. This means the magnetic induction can be expressed as curl of a vector potential

$$\boxed{\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}), \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}\left(\mathbf{r}'\right)}{|\mathbf{r}-\mathbf{r}'|}} \tag{1.8}$$

Further, two identities follow from this expression. The first one is

$$\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}(\mathbf{r})) = \partial_i \left(\epsilon_{ijk}\partial_j A_k\right) = \epsilon_{ijk}\partial_i\partial_j A_k = 0 \tag{1.9}$$

and the second one, also known as the Ampere's law, is given by

$$\begin{aligned}
\nabla \times \mathbf{B} &= \nabla \times (\nabla \times \mathbf{A}(\mathbf{r})) = \epsilon_{ijk}\partial_j \left(\epsilon_{klm}\partial_l A_m\right) = \epsilon_{ijk}\epsilon_{klm}\partial_j\partial_l A_m && (1.10)\\
&= \left(\delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}\right)\partial_j\partial_l A_m && (1.11)\\
&= \partial_i\partial_j A_j - \partial_j\partial_j A_i && (1.12)\\
&= \nabla\left(\nabla\cdot\mathbf{A}\right) - \nabla^2\mathbf{A} && (1.13)
\end{aligned}$$

Using $\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = -\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)$, $\nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = -4\pi\delta^{(3)}\left(\mathbf{r}-\mathbf{r}'\right)$ we arrive at

$$\begin{aligned}
\nabla \times \mathbf{B} &= \nabla\left(\nabla\cdot\mathbf{A}\right) - \nabla^2\mathbf{A} && (1.14)\\
&= \nabla\left(\nabla\cdot\frac{\mu_0}{4\pi}\int d^3r'\frac{\mathbf{J}\left(\mathbf{r}'\right)}{|\mathbf{r}-\mathbf{r}'|}\right) - \nabla^2\frac{\mu_0}{4\pi}\int d^3r'\frac{\mathbf{J}\left(\mathbf{r}'\right)}{|\mathbf{r}-\mathbf{r}'|} && (1.15)\\
&= \nabla\left(\frac{\mu_0}{4\pi}\int d^3r'\mathbf{J}\left(\mathbf{r}'\right)\cdot\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\right) - \frac{\mu_0}{4\pi}\int d^3r'\mathbf{J}\left(\mathbf{r}'\right)\nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) && (1.16)\\
&= \frac{\mu_0}{4\pi}\int d^3r'\mathbf{J}\left(\mathbf{r}'\right)4\pi\delta^{(3)}\left(\mathbf{r}-\mathbf{r}'\right) - \nabla\left(\frac{\mu_0}{4\pi}\int d^3r'\mathbf{J}\left(\mathbf{r}'\right)\cdot\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\right) && (1.17)\\
&= \mu_0\mathbf{J}(\mathbf{r}) - \nabla\left(\frac{\mu_0}{4\pi}\int d^3r'\left[\nabla'\cdot\left(\frac{\mathbf{J}\left(\mathbf{r}'\right)}{|\mathbf{r}-\mathbf{r}'|}\right) - \frac{\nabla'\cdot\mathbf{J}\left(\mathbf{r}'\right)}{|\mathbf{r}-\mathbf{r}'|}\right]\right) && (1.18)\\
&= \mu_0\mathbf{J}(\mathbf{r}) + \frac{\mu_0}{4\pi}\nabla\left(\int d^3r'\frac{\nabla'\cdot\mathbf{J}\left(\mathbf{r}'\right)}{|\mathbf{r}-\mathbf{r}'|}\right) && (1.19)
\end{aligned}$$

For steady-state currents $\nabla'\cdot\mathbf{J}\left(\mathbf{r}'\right) = 0$, we get Ampere's law

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0\mathbf{J}(\mathbf{r}) \tag{1.20}$$

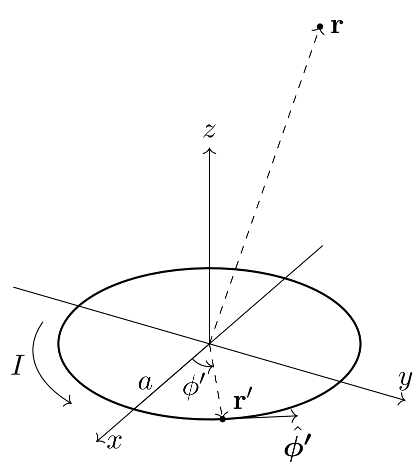
To conclude, starting with the Biot-Savart law, we obtain two Maxwell equations for steady-state currents

$$\boxed{\begin{cases}\nabla\cdot\mathbf{B} = 0 & \text{source-free}\\ \nabla\times\mathbf{B}(\mathbf{r}) = \mu_0\mathbf{J}(\mathbf{r}) & \text{Ampere's law}\end{cases}} \tag{1.21}$$

1.2 A circular current loop

1.2.1 Setup and the cylindrical way

Vector potential A circular current loop of radius a carrying a current I lies in the xy-plane with its center at the origin.



The current density in cylindrical coordinates is

$$\mathbf{J}\left(\mathbf{r}'\right) = I\delta\left(\rho' - a\right)\delta\left(z'\right)\hat{\phi}' \tag{1.22}$$

where $\hat{\phi}' = -\sin\phi'\hat{\mathbf{x}} + \cos\phi'\hat{\mathbf{y}}$ is the tangential unit vector in the direction of current. Then the vector potential

$$\begin{aligned}
\mathbf{A}\left(\mathbf{r}\right) &= \frac{\mu_0}{4\pi}\int d^3r' \frac{\mathbf{J}\left(\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|} && (1.23)\\
&= \frac{\mu_0}{4\pi}\int_0^\infty d\rho'\rho' \int_{-\infty}^{\infty} dz' \int_0^{2\pi} d\phi' \frac{I\delta\left(\rho'-a\right)\delta\left(z'\right)\hat{\phi}'}{\left|\mathbf{r}-\mathbf{r}'\right|} && (1.24)\\
&= \frac{\mu_0 I a}{4\pi}\int_0^{2\pi} d\phi' \frac{-\sin\phi'\hat{\mathbf{x}} + \cos\phi'\hat{\mathbf{y}}}{\sqrt{\rho^2 + a^2 - 2\rho a\cos\left(\phi-\phi'\right) + z^2}} && (1.25)
\end{aligned}$$

We expand the denominator in the integrand using the addition theorem for Bessel functions [3, Eq. (4.32)]

$$\frac{1}{\sqrt{\rho^2 + a^2 - 2\rho a\cos\left(\phi-\phi'\right) + z^2}} = \sum_{m=-\infty}^{\infty}\int_0^\infty dk J_m\left(k\rho\right) J_m\left(ka\right) e^{im\left(\phi-\phi'\right)} e^{-k|z|} \tag{1.26}$$

Plugging this back gives

$$\begin{aligned}
\mathbf{A}\left(\mathbf{r}\right) &= \frac{\mu_0 I a}{4\pi}\int_0^{2\pi} d\phi' \left(-\sin\phi'\hat{\mathbf{x}} + \cos\phi'\hat{\mathbf{y}}\right)\sum_{m=-\infty}^{\infty}\int_0^\infty dk J_m\left(k\rho\right) J_m\left(ka\right) e^{im\left(\phi-\phi'\right)} e^{-k|z|} && (1.27)\\
&= \frac{\mu_0 I a}{4\pi}\sum_{m=-\infty}^{\infty}\int_0^{2\pi} d\phi' \left(-\sin\phi'\hat{\mathbf{x}} + \cos\phi'\hat{\mathbf{y}}\right) e^{im\left(\phi-\phi'\right)}\int_0^\infty dk J_m\left(k\rho\right) J_m\left(ka\right) e^{-k|z|} && (1.28)\\
&= \frac{\mu_0 I a}{4\pi}\sum_{m=-\infty}^{\infty} 2\pi\left(\frac{i\delta_{m,1} - i\delta_{m,-1}}{2}\hat{\mathbf{x}} + \frac{\delta_{m,1} + \delta_{m,-1}}{2}\hat{\mathbf{y}}\right) e^{im\phi}\int_0^\infty dk J_m\left(k\rho\right) J_m\left(ka\right) e^{-k|z|} && (1.29)\\
&= \frac{\mu_0 I a}{2}\left(\frac{ie^{i\phi} - ie^{-i\phi}}{2}\hat{\mathbf{x}} + \frac{e^{i\phi} + e^{-i\phi}}{2}\hat{\mathbf{y}}\right)\int_0^\infty dk J_1\left(k\rho\right) J_1\left(ka\right) e^{-k|z|} && (1.30)\\
&= \frac{\mu_0 I a}{2}\left(-\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}\right)\int_0^\infty dk J_1\left(k\rho\right) J_1\left(ka\right) e^{-k|z|} && (1.31)
\end{aligned}$$

where we have used $J_{-m}\left(x\right) = \left(-1\right)^m J_m\left(x\right)$. Since $\hat{\phi} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}$,

$$\boxed{\mathbf{A}\left(\mathbf{r}\right) = \frac{\mu_0 I a\hat{\phi}}{2}\int_0^\infty dk J_1\left(k\rho\right) J_1\left(ka\right) e^{-k|z|}} \tag{1.32}$$

A byproduct from this derivation is

$$\sum_{m=-\infty}^{\infty}\int_0^{2\pi} d\phi' \left(-\sin\phi'\hat{\mathbf{x}} + \cos\phi'\hat{\mathbf{y}}\right) e^{im\left(\phi-\phi'\right)} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}} = \hat{\phi} \tag{1.33}$$

Magnetic induction From $\mathbf{B} = \nabla\times\mathbf{A}$, we get

$$\begin{cases} B_\rho = \frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z} \\ B_\phi = \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho} \\ B_z = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho A_\phi\right) - \frac{1}{\rho}\frac{\partial A_\rho}{\partial\phi} \end{cases} \tag{1.34}$$

Using

$$A_\phi = \frac{\mu_0 I a}{2}\int_0^\infty dk J_1\left(k\rho\right) J_1\left(ka\right) e^{-k|z|}, \quad A_\rho = A_z = 0, \tag{1.35}$$

we find

$$B_\rho = -\frac{\partial A_\phi}{\partial z} = -\frac{\partial}{\partial z}\frac{\mu_0 I a}{2}\int_0^\infty dk J_1(k\rho)\, J_1(ka)\, e^{-k|z|} \tag{1.36}$$

$$= \frac{\mu_0 I a}{2}\mathrm{sgn}(z)\int_0^\infty dk k J_1(k\rho)\, J_1(ka)\, e^{-k|z|} \tag{1.37}$$

$$B_\phi = 0 \tag{1.38}$$

$$B_z = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\phi) = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\mu_0 I a}{2}\int_0^\infty dk J_1(k\rho)\, J_1(ka)\, e^{-k|z|}\right) \tag{1.39}$$

$$= \frac{\mu_0 I a}{2}\left[\int_0^\infty dk k\frac{\partial J_1(k\rho)}{\partial k\rho} J_1(ka)\, e^{-k|z|} + \int_0^\infty dk k\frac{J_1(k\rho)}{k\rho} J_1(ka)\, e^{-k|z|}\right] \tag{1.40}$$

$$= \frac{\mu_0 I a}{2}\int_0^\infty dk k\left[J_1'(k\rho) + \frac{J_1(k\rho)}{k\rho}\right] J_1(ka)\, e^{-k|z|} \tag{1.41}$$

$$= \frac{\mu_0 I a}{2}\int_0^\infty dk k\left[J_0(k\rho) - \frac{J_1(k\rho)}{k\rho} + \frac{J_1(k\rho)}{k\rho}\right] J_1(ka)\, e^{-k|z|} \tag{1.42}$$

$$= \frac{\mu_0 I a}{2}\int_0^\infty dk k J_0(k\rho) J_1(ka)\, e^{-k|z|} \tag{1.43}$$

where we have used the recurrence relation of the Bessel functions ([1, Eq. 9.1.27])

$$J_1'(x) = J_0(x) - \frac{J_1(x)}{x} \tag{1.44}$$

To conclude

$$\boxed{\begin{cases} B_\rho = \frac{\mu_0 I a}{2}\mathrm{sgn}(z)\int_0^\infty dk k J_1(k\rho)\, J_1(ka)\, e^{-k|z|} \\ B_\phi = 0 \\ B_z = \frac{\mu_0 I a}{2}\int_0^\infty dk k J_0(k\rho) J_1(ka)\, e^{-k|z|} \end{cases}} \tag{1.45}$$

On the axis $\rho = 0$, since $J_1(0) = 0$ and $J_0(0) = 1$, we have $B_\rho = 0$ and

$$B_z|_{\rho=0} = \frac{\mu_0 I a}{2}\int_0^\infty dk k J_1(ka)\, e^{-k|z|} \tag{1.46}$$

Note $J_1(x) = -J_0'(x)$,

$$J_1(ka) = -J_0'(ka) = -\frac{\partial J_0(ka)}{\partial (ka)} = -\frac{1}{k}\frac{\partial J_0(ka)}{\partial a} \tag{1.47}$$

so

$$B_z|_{\rho=0} = \frac{\mu_0 I a}{2}\int_0^\infty dk k J_1(ka)\, e^{-k|z|} = -\frac{\mu_0 I a}{2}\int_0^\infty dk k\frac{1}{k}\frac{\partial J_0(ka)}{\partial a} e^{-k|z|} \tag{1.48}$$

$$= -\frac{\mu_0 I a}{2}\frac{\partial}{\partial a}\int_0^\infty dk J_0(ka) e^{-k|z|} \tag{1.49}$$

$$= -\frac{\mu_0 I a}{2}\frac{\partial}{\partial a}\frac{1}{\sqrt{a^2+z^2}} \tag{1.50}$$

$$= \frac{\mu_0 I a^2}{2\left(a^2+z^2\right)^{3/2}} \tag{1.51}$$

$$\boxed{B_z|_{\rho=0} = \frac{\mu_0 I}{2}\frac{a^2}{\left(a^2+z^2\right)^{3/2}}, \quad \text{on axis}} \tag{1.52}$$

1.2.2 The spherical way

Vector potential The current density of the loop in spherical coordinates is

$$\mathbf{J}\left(\mathbf{r}'\right) = I\frac{\delta\left(r'-a\right)}{a}\delta\left(\cos\theta'\right)\hat{\phi}' \tag{1.53}$$

so

$$\begin{aligned}\mathbf{A}\left(\mathbf{r}\right) &= \frac{\mu_0}{4\pi}\int d^3r'\frac{\mathbf{J}\left(\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|} &(1.54)\\ &= \frac{\mu_0}{4\pi}\int_0^\infty dr' r'^2\int_0^\pi d\theta'\sin\theta'\int_0^{2\pi} d\phi'\frac{I\frac{\delta(r'-a)}{a}\delta\left(\cos\theta'\right)\hat{\phi}'}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} &(1.55)\end{aligned}$$

where γ is the angle between $\mathbf{r}$ and $\mathbf{r}'$ given by [1]

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\left(\phi-\phi'\right) \tag{1.56}$$

Then for $r > a$, the vector potential

$$\begin{aligned}\mathbf{A}\left(\mathbf{r}\right) &= \frac{\mu_0 I a}{4\pi}\int_0^{2\pi} d\phi'\frac{\left(-\sin\phi'\hat{\mathbf{x}}+\cos\phi'\hat{\mathbf{y}}\right)}{\sqrt{r^2+a^2-2ra\sin\theta\cos\left(\phi-\phi'\right)}} &(1.57)\\ &= \frac{\mu_0 I a}{4\pi}\int_0^{2\pi} d\phi'\left(-\sin\phi'\hat{\mathbf{x}}+\cos\phi'\hat{\mathbf{y}}\right)\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*\left(\theta'=\frac{\pi}{2},\phi'\right)Y_{lm}\left(\theta,\phi\right) &(1.58)\\ &= \frac{\mu_0 I a}{4\pi}2\pi\hat{\phi}\sum_{l=1}^{\infty}\frac{(l-1)!}{(l+1)!}\frac{a^l}{r^{l+1}}P_l^1\left(0\right)P_l^1\left(\cos\theta\right) &(1.59)\\ &= \frac{\mu_0 I a}{2}\hat{\phi}\sum_{l=1}^{\infty}\frac{1}{l(l+1)}\frac{a^l}{r^{l+1}}P_l^1\left(0\right)P_l^1\left(\cos\theta\right) &(1.60)\end{aligned}$$

Now we evaluate $P_l^1(0)$. Note $P_l^m(x) \equiv (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_l(x)$,

$$P_l^1(x) = -\left(1-x^2\right)^{1/2}\frac{d}{dx}P_l(x) \tag{1.61}$$

From the generating function

$$\frac{1}{\sqrt{1+t^2-2tx}} = \sum_{l=0}^{\infty}t^l P_l(x) \tag{1.62}$$

take a derivative w.r.t. x

$$\frac{\partial}{\partial x}\frac{1}{\sqrt{1+t^2-2tx}} = \sum_{l=0}^{\infty}t^l P_l'(x) \tag{1.63}$$

At $x=0$, the left hand side goes to

$$\begin{aligned}\frac{\partial}{\partial x}\frac{1}{\sqrt{1+t^2-2tx}}\bigg|_{x=0} &= \frac{t}{\left(1+t^2-2tx\right)^{3/2}}\bigg|_{x=0} = \frac{t}{\left(1+t^2\right)^{3/2}} = t\sum_{n=0}^{\infty}\binom{-3/2}{n}t^{2n} &(1.64)\\ &= \sum_{n=0}^{\infty}\binom{-3/2}{n}t^{2n+1} = \sum_{n=0}^{\infty}\frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2n+1}{2}\right)}{n!}t^{2n+1} &(1.65)\\ &= \sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)!!}{2^n n!}t^{2n+1} &(1.66)\end{aligned}$$

[1] This can be seen from the dot product identity $\mathbf{r}\cdot\mathbf{r}' = rr'\cos\gamma = xx'+yy'+zz' = rr'\left[\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\left(\phi-\phi'\right)\right]$

and the right hand side is

$$\sum_{l=0}^{\infty} t^l P_l'(0) \tag{1.67}$$

Matching coefficients of t^l gives

$$P_l'(0) = \begin{cases} 0 & l \text{ odd} \\ (-1)^n \frac{(2n+1)!!}{2^n n!} & l = 2m+1 \end{cases} \tag{1.68}$$

then

$$P_l^1(0) = -P_l'(0) = \begin{cases} 0 & l \text{ odd} \\ (-1)^{n+1} \frac{(2n+1)!!}{2^n n!} & l = 2m+1 \end{cases} \tag{1.69}$$

So the vector potential becomes

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I a}{2} \hat{\phi} \sum_{l=1}^{\infty} \frac{1}{l(l+1)} \frac{a^l}{r^{l+1}} P_l^1(0) P_l^1(\cos\theta) \tag{1.70}$$

$$= \frac{\mu_0 I a}{2} \hat{\phi} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+1)!!}{2^n n! (2n+1)(2n+2)} \frac{a^{2n+1}}{r^{2n+2}} P_{2n+1}^1(\cos\theta) \tag{1.71}$$

$$= -\frac{\mu_0 I a}{4} \hat{\phi} \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{2^n n! (n+1)} \frac{a^{2n+1}}{r^{2n+2}} P_{2n+1}^1(\cos\theta) \tag{1.72}$$

$$\boxed{\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 I a}{4} \hat{\phi} \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{2^n n! (n+1)} \frac{a^{2n+1}}{r^{2n+2}} P_{2n+1}^1(\cos\theta)} \tag{1.73}$$

Magnetic induction The magnetic induction $\mathbf{B} = \nabla \times \mathbf{A}$ is

$$\begin{cases} B_r = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right] \\ B_\theta = \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right] \\ B_\phi = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right] \end{cases} \tag{1.74}$$

so

$$B_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi), \quad B_\theta = -\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi), \quad B_\phi = 0 \tag{1.75}$$

provided that $A_r = A_\theta = 0$. With Eq. (1.73)

$$B_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) = -\frac{\mu_0 I a}{4}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{2^n n! (n+1)} \frac{a^{2n+1}}{r^{2n+2}} P_{2n+1}^1(\cos\theta)\right) \tag{1.76}$$

$$= \frac{\mu_0 I a}{4r} \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{2^n n! (n+1)} \frac{a^{2n+1}}{r^{2n+2}} \frac{\partial}{\partial\cos\theta}\left(\sin\theta P_{2n+1}^1(\cos\theta)\right) \tag{1.77}$$

Let $x = \cos\theta$,

$$\frac{\partial}{\partial\cos\theta}\left(\sin\theta P_l^1(\cos\theta)\right) = \frac{d}{dx}\left(\sqrt{1-x^2}P_l^1(x)\right) = \frac{d}{dx}\left[-\left(1-x^2\right)P_l'(x)\right] \tag{1.78}$$

$$= 2xP_l'(x) - (1-x^2)P_l''(x) \tag{1.79}$$

Note the Legendre polynomials satisfy the ODE [1, Eq. 8.1.1]

$$(1-x^2)P_l''(x) - 2xP_l'(x) + l(l+1)P_l(x) = 0 \tag{1.80}$$

then

$$\frac{\partial}{\partial \cos\theta}\left(\sin\theta P_l^1(\cos\theta)\right) = 2xP_l'(x) - (1-x^2)P_l''(x) = l(l+1)P_l(x) \tag{1.81}$$

$$= l(l+1)P_l(\cos\theta) \tag{1.82}$$

and

$$B_r = \frac{\mu_0 I a}{4r}\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{2^n n!(n+1)}\frac{a^{2n+1}}{r^{2n+2}}(2n+1)(2n+2)P_{2n+1}(\cos\theta) \tag{1.83}$$

$$= \frac{\mu_0 I a}{2r}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)!!}{2^n n!}\frac{a^{2n+1}}{r^{2n+2}}P_{2n+1}(\cos\theta) \tag{1.84}$$

The θ-component

$$B_\theta = -\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\mu_0 I a}{4}\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{2^n n!(n+1)}\frac{a^{2n+1}}{r^{2n+2}}P_{2n+1}^1(\cos\theta)\right) \tag{1.85}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\mu_0 I a}{4}\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{2^n n!(n+1)}\frac{a^{2n+1}}{r^{2n+1}}P_{2n+1}^1(\cos\theta)\right) \tag{1.86}$$

$$= -\frac{\mu_0 I a^2}{4}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)!!}{2^n(n+1)!}\frac{a^{2n}}{r^{2n+3}}P_{2n+1}^1(\cos\theta) \tag{1.87}$$

To conclude, for $r > a$

$$\boxed{\begin{cases} B_r = \frac{\mu_0 I a^2}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)!!}{2^n n!}\frac{a^{2n}}{r^{2n+3}}P_{2n+1}(\cos\theta) \\ B_\theta = -\frac{\mu_0 I a^2}{4}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)!!}{2^n(n+1)!}\frac{a^{2n}}{r^{2n+3}}P_{2n+1}^1(\cos\theta) \\ B_\phi = 0 \end{cases}} \tag{1.88}$$

2 Magnetic multipole expansion

For a localized current distribution $\mathbf{J}(\mathbf{r}')$, the vector potential at a point $\mathbf{r}$ far away from $\mathbf{r}'$ is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r'\,\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{2.1}$$

Assuming $|\mathbf{r}| \gg |\mathbf{r}'|$, the denominator can be expanded

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} = \frac{1}{r}\frac{1}{\sqrt{1+\frac{r'^2}{r^2}-2\frac{r'}{r}\cos\gamma}} \tag{2.2}$$

$$= \frac{1}{r}\left(1+\frac{r'}{r}\cos\gamma + O\left(r'^2/r^2\right)\right) \tag{2.3}$$

$$= \frac{1}{r}+\frac{rr'\cos\gamma}{r^3}+\cdots \tag{2.4}$$

$$= \frac{1}{|\mathbf{r}|}+\frac{\mathbf{r}\cdot\mathbf{r}'}{|\mathbf{r}|^3}+\cdots \tag{2.5}$$

Then

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi}\left[\frac{1}{|\mathbf{r}|}\int d^3r'\,J_i(\mathbf{r}') + \frac{\mathbf{r}}{|\mathbf{r}|^3}\cdot\int d^3r'\,\mathbf{r}'J_i(\mathbf{r}')+\cdots\right] \tag{2.6}$$

2.1 Monopole

Note on one hand Stokes theorem gives

$$\int d^3r'\nabla'\cdot\left(r_i'\mathbf{J}\right)=\int d^2A'\hat{n}'\cdot\left(r_i'\mathbf{J}\right)\to 0 \tag{2.7}$$

and on the other hand with the Leibniz rule

$$\int d^3r'\nabla'\cdot\left(r_i'\mathbf{J}\right)=\int d^3r'\left(r_i'\nabla'\cdot\mathbf{J}+\nabla'r_i'\cdot\mathbf{J}\right)=\int d^3r'J_i\left(r'\right)=0 \tag{2.8}$$

where we have assumed steady current $\nabla'\cdot\mathbf{J}=0$. So the monopole term in the vector potential vanishes

$$\frac{\mu_0}{4\pi}\frac{1}{|\mathbf{r}|}\int d^3r'J_i\left(\mathbf{r}'\right)=0 \tag{2.9}$$

2.2 Dipole

Note

$$\begin{aligned}\int d^3r'\partial_k'\left(r_i'r_j'J_k\right)&=\int d^3r'\left[\left(\partial_k'r_i'\right)r_j'J_k+r_i'\left(\partial_k'r_j'\right)J_k+r_i'r_j'\partial_k'J_k\right] &(2.10)\\&=\int d^3r'\left(r_j'J_i+r_i'J_j+r_i'r_j'\nabla'\cdot\mathbf{J}\right) &(2.11)\\&=0 &(2.12)\end{aligned}$$

then for steady currents

$$0=\int d^3r'\left(r_j'J_i+r_i'J_j\right) \tag{2.13}$$

Now with this identity we may rewrite the integral in the dipole term as

$$\begin{aligned}\mathbf{r}\int d^3r'\mathbf{r}'J_i\left(\mathbf{r}'\right)=r_j\int d^3r'r_j'J_i&=\frac{1}{2}r_j\int d^3r'\left(r_j'J_i-r_i'J_j\right) &(2.14)\\&=-\frac{1}{2}r_j\int d^3r'\epsilon_{ijk}\left(\mathbf{r}'\times\mathbf{J}\right)_k &(2.15)\\&=-\frac{1}{2}\left(\mathbf{r}\times\int d^3r'\mathbf{r}'\times\mathbf{J}\right)_i &(2.16)\\&=\left[\left(\frac{1}{2}\int d^3r'\mathbf{r}'\times\mathbf{J}\right)\times\mathbf{r}\right]_i &(2.17)\end{aligned}$$

If we define the magnetic moment

$$\mathbf{m}=\frac{1}{2}\int d^3r'\mathbf{r}'\times\mathbf{J} \tag{2.18}$$

then the dipole term is

$$\frac{\mu_0}{4\pi}\frac{\mathbf{r}}{|\mathbf{r}|^3}\cdot\int d^3r'\mathbf{r}'\mathbf{J}\left(\mathbf{r}'\right)=\frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{|\mathbf{r}|^3} \tag{2.19}$$

Finally, the vector potential up to dipole is

$$\mathbf{A}\left(\mathbf{r}\right)=\frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{|\mathbf{r}|^3},\quad \mathbf{m}=\frac{1}{2}\int d^3r'\mathbf{r}'\times\mathbf{J}\left(\mathbf{r}'\right) \tag{2.20}$$

2.3 Force and energy

3 Magnetization

3.1 Theory

The current density in magnetized matter comprises of two parts, conducting currents induced by moving charges and nuclear moments due to quantum-mechanical effects. Define the magnetization of a chunk of magnetized matter as

$$\mathbf{m}\left(\mathbf{r}'\right) \equiv dr'^3 \mathbf{M}\left(\mathbf{r}'\right) \tag{3.1}$$

The vector potential produced by these moments is

$$\mathbf{A}\left(\mathbf{r}\right) = \frac{\mu_0}{4\pi} \sum_{\mathbf{r}'} \frac{\mathbf{m}\left(\mathbf{r}'\right) \times \left(\mathbf{r} - \mathbf{r}'\right)}{\left|\mathbf{r} - \mathbf{r}'\right|^3} = \frac{\mu_0}{4\pi} \int d^3 r' \frac{\mathbf{M}\left(\mathbf{r}'\right) \times \left(\mathbf{r} - \mathbf{r}'\right)}{\left|\mathbf{r} - \mathbf{r}'\right|^3} \tag{3.2}$$

Note

$$\frac{\mathbf{r} - \mathbf{r}'}{\left|\mathbf{r} - \mathbf{r}'\right|^3} = -\nabla \left(\frac{1}{\left|\mathbf{r} - \mathbf{r}'\right|}\right) = \nabla' \left(\frac{1}{\left|\mathbf{r} - \mathbf{r}'\right|}\right) \tag{3.3}$$

so

$$A_i\left(\mathbf{r}\right) = \frac{\mu_0}{4\pi} \left[\int d^3 r' \mathbf{M}\left(\mathbf{r}'\right) \times \nabla' \left(\frac{1}{\left|\mathbf{r} - \mathbf{r}'\right|}\right)\right]_i \tag{3.4}$$

$$= \frac{\mu_0}{4\pi} \epsilon_{ijk} \int d^3 r' M_j \partial'_k \left(\frac{1}{\left|\mathbf{r} - \mathbf{r}'\right|}\right) \tag{3.5}$$

$$= \frac{\mu_0}{4\pi} \epsilon_{ijk} \int d^3 r' \left[\partial'_k \left(\frac{M_j\left(\mathbf{r}'\right)}{\left|\mathbf{r} - \mathbf{r}'\right|}\right) - \frac{\partial'_k M_j\left(\mathbf{r}'\right)}{\left|\mathbf{r} - \mathbf{r}'\right|}\right] \tag{3.6}$$

$$= \frac{\mu_0}{4\pi} \left(\epsilon_{ijk} \int dA' \frac{\hat{n}'_k M_j\left(\mathbf{r}'\right)}{\left|\mathbf{r} - \mathbf{r}'\right|} - \epsilon_{ijk} \int d^3 r' \frac{\partial'_k M_j\left(\mathbf{r}'\right)}{\left|\mathbf{r} - \mathbf{r}'\right|}\right) \tag{3.7}$$

$$= \left[\frac{\mu_0}{4\pi} \oint dA' \frac{-\hat{n'} \times \mathbf{M}\left(\mathbf{r}'\right)}{\left|\mathbf{r} - \mathbf{r}'\right|} + \frac{\mu_0}{4\pi} \int d^3 r' \frac{\nabla' \times \mathbf{M}\left(\mathbf{r}'\right)}{\left|\mathbf{r} - \mathbf{r}'\right|}\right]_i \tag{3.8}$$

$$\mathbf{A}\left(\mathbf{r}\right) = \frac{\mu_0}{4\pi} \oint dA' \frac{-\hat{n'} \times \mathbf{M}\left(\mathbf{r}'\right)}{\left|\mathbf{r} - \mathbf{r}'\right|} + \frac{\mu_0}{4\pi} \int d^3 r' \frac{\nabla' \times \mathbf{M}\left(\mathbf{r}'\right)}{\left|\mathbf{r} - \mathbf{r}'\right|} \tag{3.9}$$

Define the equivalent magnetic surface current density

$$\mathbf{j}_M\left(\mathbf{r}'\right) \equiv -\hat{n'} \times \mathbf{M}\left(\mathbf{r}'\right) \tag{3.10}$$

and the equivalent magnetization current

$$\mathbf{J}_M\left(\mathbf{r}'\right) \equiv \nabla' \times \mathbf{M}\left(\mathbf{r}'\right) \tag{3.11}$$

Then

$$\boxed{\mathbf{A}\left(\mathbf{r}\right) = \frac{\mu_0}{4\pi} \int d^3 r' \frac{\mathbf{J}_M\left(\mathbf{r}'\right)}{\left|\mathbf{r} - \mathbf{r}'\right|} + \frac{\mu_0}{4\pi} \oint dA' \frac{\mathbf{j}_M\left(\mathbf{r}'\right)}{\left|\mathbf{r} - \mathbf{r}'\right|}} \tag{3.12}$$

Now the entire magnetic induction is created from both conduction currents and magnetization

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J}_{\text{cond}} + \mathbf{J}_M\right) \tag{3.13}$$

in which $\mathbf{J}_{\text{cond}}$ is the conducting currents due to moving charges, and $\mathbf{J}_M$ are fictitious currents that is an equivalent description of magnetic field created by nuclear moments in matter. Let $\mathbf{H}$ denote the field created only by conduction currents

$$\nabla \times \mathbf{H} = \mathbf{J}_{\text{cond}} \tag{3.14}$$

and call it the magnetic fields, then

$$\nabla \times \mathbf{B} = \mu_0 \nabla \times \mathbf{H} + \mu_0 \mathbf{J}_M = \mu_0 \nabla \times (\mathbf{H} + \mathbf{M}) \tag{3.15}$$

which leads to

$$\boxed{\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M})} \tag{3.16}$$

Ferromagnetic For ferromagnetic material there is a permanent magnetization $\mathbf{M}$, from which we compute the equivalent magnetization charges and currents, and then we compute $\mathbf{B}$ and $\mathbf{H}$ from these sources along with any induction currents present.

Paramagnetic For permeable matter magnetized by external inductions we first compute $\mathbf{H}$

$$\nabla \times \mathbf{H} = \mu_0 \mathbf{J}_{\text{cond}}, \qquad \oint d\mathbf{l} \cdot \mathbf{H} = I_{\text{cond}} \tag{3.17}$$

then find $\mathbf{B}$ from $\mathbf{B} = \mu \mathbf{H}$ for linear magnetized materials. The magnetization $\mathbf{M}$ is then obtained from the constitutive relation

$$\mathbf{B} = \mu \mathbf{H} = \mu_0 (\mathbf{H} + \mathbf{M}) \tag{3.18}$$

3.2 Magnetic scalar potential

In some special cases we may define a magnetic scalar potential Φ_M as follows. For example, in current-free region

$$\nabla \times \mathbf{H} = \mathbf{J}_{\text{cond}} = 0 \tag{3.19}$$

then $\mathbf{H}$ can be expressed as the gradient of a scalar potential

$$\mathbf{H} = -\nabla \Phi_M \tag{3.20}$$

The divergence-free equation $\nabla \cdot \mathbf{B} = 0$ leads to

$$0 = \nabla \cdot \mathbf{B} = \mu_0 \nabla \cdot (\mathbf{H} + \mathbf{M}) = \mu_0 \nabla \cdot (-\nabla \Phi_M + \mathbf{M}) \tag{3.21}$$

$$\nabla^2 \Phi_M = \nabla \cdot \mathbf{M} \tag{3.22}$$

If we introduce a magnetic charge density

$$\rho_M \equiv -\nabla \cdot \mathbf{M} \tag{3.23}$$

then Φ_M satisfies Poisson equation

$$\nabla^2 \Phi_M = \nabla \cdot \mathbf{M} = -\rho_M \tag{3.24}$$

The general solution is

$$\Phi_M (\mathbf{r}) = \frac{1}{4\pi} \int d^3 r' \frac{\rho_M (\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{4\pi} \int d^3 r' \frac{-\nabla' \cdot \mathbf{M} (\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \tag{3.25}$$

If there is discontinuity in $\mathbf{M}$ (for example, crossing boundaries of a ferromagnetic material) then we need to add a surface charge term to the magnetic scalar potential

$$\Phi_M (\mathbf{r}) = \frac{1}{4\pi} \int_V d^3 r' \frac{-\nabla' \cdot \mathbf{M} (\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \frac{1}{4\pi} \oint_{\partial V} d^2 A' \frac{\hat{\mathbf{n}}' \cdot \mathbf{M} (\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \tag{3.26}$$

where we have introduced an effective magnetic surface-charge density $\sigma_M \equiv \hat{\mathbf{n}} \cdot \mathbf{M}$.

3.3 Boundary conditions on permeable surfaces

On a permeable interface S with permeabilities μ_1 and μ_2, Ampere's law gives

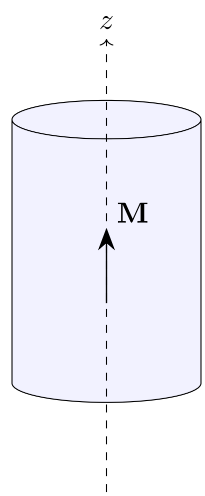
$$\oint d\mathbf{l}\cdot\mathbf{H} = I_{\text{cond},\perp} \quad \rightarrow \quad (\mathbf{H}_1 - \mathbf{H}_2)\cdot\mathbf{t}|_S = \mathbf{K}_{\text{cond}} \tag{3.27}$$

where $\mathbf{t}$ is the tangent on the surface and $\mathbf{K}_{\text{cond}}$ is the conducting current density on the interface. And Gauss's law implies

$$\oint d^2A\hat{\mathbf{n}}\cdot\mathbf{B} = 0 \quad \rightarrow \quad (\mathbf{B}_1 - \mathbf{B}_2)\cdot\mathbf{n}|_S = 0 \tag{3.28}$$

3.4 Examples

Uniformly magnetized cylinder A permanent magnet in the shape of a right circular cylinder coaxial with the z-axis with its center at the origin. It has radius a and length l, and is permanently magnetized with $\mathbf{M} = M_0\hat{\mathbf{z}}$. There are no conduction currents everywhere, find $\mathbf{B}$ at $(0,0,z)$ where $|z| > l$.



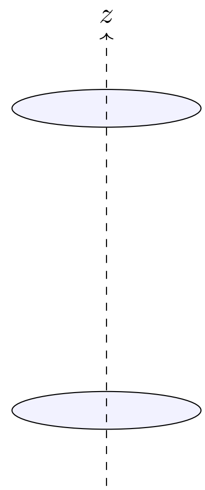
Because there are no conduction currents we may invoke the magnetic potential way with

$$\Phi_M = \frac{1}{4\pi}\int_V d^3r'\frac{-\nabla'\cdot\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \frac{1}{4\pi}\oint_{\partial V} d^2A'\frac{\hat{\mathbf{n}}'\cdot\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{3.29}$$

Inside the cylinder $\nabla'\cdot\mathbf{M}(\mathbf{r}') = 0$ and on the surfaces at $z = \pm l/2$ we have $\hat{\mathbf{n}} = \pm\hat{\mathbf{z}}$, so

$$\sigma_M|_{z=\pm l/2} = \hat{\mathbf{n}}\cdot\mathbf{M}(\mathbf{r})|_{z=\pm l/2} = \pm M_0 \tag{3.30}$$

Then this problem is effectively isomorphic to the electrostatic problem of finding the electric potential due to two uniformly distributed surface charge disks at $z = \pm l/2$.



That means

$$\Phi_M = \frac{1}{4\pi}\int_0^a \rho d\rho d\phi \frac{M_0}{\sqrt{\rho^2 + (z-l)^2}} - \frac{1}{4\pi}\int_0^a \rho d\rho d\phi \frac{M_0}{\sqrt{\rho^2 + (z-l)^2}} \tag{3.31}$$

$$= \frac{M_0}{2}\left(\sqrt{a^2 + (z-l)^2} - \sqrt{a^2 + (z+l)^2} + 2l\right) \tag{3.32}$$

The magnetic field

$$H_z = -\partial_z \Phi_M = \frac{M_0}{2}\left(\frac{z-l}{\sqrt{a^2 + (z-l)^2}} - \frac{z+l}{\sqrt{a^2 + (z-l)^2}}\right) \tag{3.33}$$

and the magnetic induction

$$B_z = \mu_0 H = \frac{\mu_0 M_0}{2}\left(\frac{z-l}{\sqrt{a^2 + (z-l)^2}} - \frac{z+l}{\sqrt{a^2 + (z-l)^2}}\right) = \frac{\mu_0 M_0}{2}\left(\cos\theta_1 + \cos\theta_2\right) \tag{3.34}$$

Rotating dielectric sphere A dielectric sphere with ϵ rotates about the z-axis at rate ω in an otherwise constant electric field $\mathbf{E} = E_0\hat{\mathbf{x}}$. Compute $\mathbf{H}$.

The electric potential outside the sphere written in spherical coordinates is

$$\Phi(\mathbf{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm} r^l + B_{lm} r^{-l-1}\right) Y_{l,m}(\theta, \phi) \tag{3.35}$$

At infinity

$$\lim_{r\to\infty}\Phi(\mathbf{r}) = -E_0 x = -E_0 r\sin\theta\cos\phi = E_0 r\sqrt{\frac{2\pi}{3}}\left(Y_{1,1} - Y_{1,-1}\right) \tag{3.36}$$

giving

$$A_{1,1} = E_0\sqrt{\frac{2\pi}{3}}, \quad A_{1,-1} = -E_0\sqrt{\frac{2\pi}{3}} \tag{3.37}$$

so

$$\Phi_{\text{out}}(\mathbf{r}) = -E_0 r\sin\theta\cos\phi + \sum_{m=-1}^{1}\frac{B_{1,m}}{r^2} Y_{1,m}(\theta,\phi) \tag{3.38}$$

$$= \left(A_{1,1} r + \frac{B_{1,1}}{r^2}\right) Y_{1,1} + \left(A_{1,-1} r + \frac{B_{1,-1}}{r^2}\right) Y_{1,-1} \tag{3.39}$$

Inside the sphere

$$\Phi_{\text{in}}(\mathbf{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} C_{lm} r^l Y_{l,m}(\theta,\phi) \tag{3.40}$$

At the boundary $r=a$, the tangential component of the electric field and the normal component of the electric displacement are continuous, i.e.,

$$(\mathbf{E}_{\text{in}} - \mathbf{E}_{\text{out}}) \cdot \hat{\mathbf{t}}\big|_{r=a} = 0, \quad (\mathbf{D}_{\text{in}} - \mathbf{D}_{\text{out}}) \cdot \hat{\mathbf{n}}\big|_{r=a} = 0 \tag{3.41}$$

with $\hat{\mathbf{t}} = \hat{\theta}$ and $\hat{\mathbf{n}} = \hat{r}$. The continuity of $\mathbf{E}_t$ gives

$$-\left.\frac{1}{r}\frac{\partial \Phi_{\text{in}}}{\partial \theta}\right|_{r=a} = -\left.\frac{1}{r}\frac{\partial \Phi_{\text{out}}}{\partial \theta}\right|_{r=a} \tag{3.42}$$

and the continuity of $\mathbf{D}_n$ gives

$$-\epsilon \left.\frac{\partial \Phi_{\text{in}}}{\partial r}\right|_{r=a} = -\epsilon_0 \left.\frac{\partial \Phi_{\text{out}}}{\partial r}\right|_{r=a} \tag{3.43}$$

These imply $C_{lm} \neq 0$ only if $l=1, m=\pm 1$, and

$$\begin{cases} C_{1,1}a = A_{1,1}a + \frac{B_{1,1}}{a^2} \\ C_{1,-1}a = A_{1,-1}a + \frac{B_{1,-1}}{a^2} \end{cases}, \quad \begin{cases} -\epsilon C_{1,1} = -\epsilon_0 \left(A_{1,1} - \frac{2B_{1,1}}{a^3}\right) \\ -\epsilon C_{1,-1} = -\epsilon_0 \left(A_{1,-1} - \frac{2B_{1,-1}}{a^3}\right) \end{cases} \tag{3.44}$$

Solving these equations give

$$B_{1,1} = -A_{1,1}a^3 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}, \quad B_{1,-1} = -A_{1,-1}a^3 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \tag{3.45}$$

$$C_{1,1} = A_{1,1}\frac{3\epsilon_0}{\epsilon + 2\epsilon_0}, \quad C_{1,-1} = A_{1,-1}\frac{3\epsilon_0}{\epsilon + 2\epsilon_0} \tag{3.46}$$

Then

$$\Phi_{\text{in}} = r\left(C_{1,1}Y_{1,1} + C_{1,-1}Y_{1,-1}\right) = r\frac{3\epsilon_0}{\epsilon + 2\epsilon_0}\left(A_{1,1}Y_{11} + A_{1,-1}Y_{1,-1}\right) = -\frac{3\epsilon_0}{\epsilon + 2\epsilon_0}E_0 r \sin\theta\cos\phi \tag{3.47}$$

$$\Phi_{\text{out}} = \left(A_{1,1}r + \frac{B_{1,1}}{r^2}\right)Y_{1,1} + \left(A_{1,-1}r + \frac{B_{1,-1}}{r^2}\right)Y_{1,-1} \tag{3.48}$$

$$= -E_0 r \sin\theta\cos\phi - \frac{a^3}{r^2}\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}\left(A_{1,1}Y_{11} + A_{1,-1}Y_{1,-1}\right) \tag{3.49}$$

$$= -E_0 r \sin\theta\cos\phi + \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}E_0 \frac{a^3}{r^2}\sin\theta\cos\phi \tag{3.50}$$

$$\begin{cases} \Phi_{\text{in}} = -\frac{3\epsilon_0}{\epsilon + 2\epsilon_0}E_0 r \sin\theta\cos\phi \\ \Phi_{\text{out}} = -E_0 r \sin\theta\cos\phi + \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}E_0 \frac{a^3}{r^2}\sin\theta\cos\phi \end{cases} \tag{3.51}$$

The electric field inside

$$\mathbf{E}_{\text{in}} = \frac{3\epsilon_0}{\epsilon + 2\epsilon_0}\mathbf{E}_0 \tag{3.52}$$

giving the polarization

$$\mathbf{P} = (\epsilon - \epsilon_0)\,\mathbf{E} = 3\epsilon_0 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}\mathbf{E}_0 \tag{3.53}$$

and polarization charge on the surface (inside the sphere because $\mathbf{P}$ is uniform so $\rho_{\text{pol}} = \nabla \cdot \mathbf{P} = 0$)

$$\boxed{\sigma_{\text{pol}} = \mathbf{P} \cdot \hat{\mathbf{n}} = 3\epsilon_0 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}E_0 \sin\theta\cos\phi} \tag{3.54}$$

This creates a surface current density

$$\mathbf{j}_M = \sigma_{\mathrm{pol}} \omega a \sin\theta \hat{\phi} = 3\epsilon_0 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \omega a E_0 \sin\theta \cos\phi \sin\theta \hat{\phi} \tag{3.55}$$

which corresponds to an effective magnetization vector $\mathbf{M}$ such that $\mathbf{j}_M = \mathbf{M} \times \hat{\mathbf{n}}$. We deduce that at the surface

$$\mathbf{M} = M\hat{\mathbf{z}}, \quad M = 3\omega a \epsilon_0 E_0 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \sin\theta \cos\phi \tag{3.56}$$

since $\hat{\mathbf{z}} \times \hat{\mathbf{n}} = \sin\theta \hat{\phi}$. With this effective magnetization the effective monopole density at the surface

$$\sigma_M = \mathbf{M} \cdot \hat{\mathbf{n}} = 3\omega a \epsilon_0 E_0 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \sin\theta \cos\theta \cos\phi \tag{3.57}$$

giving magnetic potential outside (rf. Eq. (3.26))

$$\Phi_M = \frac{1}{4\pi} \int_S d^2A' \frac{\sigma_M(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} = \frac{a^2}{4\pi} 3\omega a \epsilon_0 E_0 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \int_S d\Omega' \frac{\sin\theta' \cos\theta' \cos\phi'}{|\mathbf{r} - \mathbf{r}'|} \tag{3.58}$$

The integral can be evaluated using the spherical harmonics addition theorem [3]

$$\int_S d\Omega' \frac{\sin\theta' \cos\theta' \cos\phi'}{|\mathbf{r} - \mathbf{r}'|} = \int_S d\Omega' \sin\theta' \cos\theta' \cos\phi' \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi) \tag{3.59}$$

$$= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta, \phi) \int_S d\Omega' \sin\theta' \cos\theta' \cos\phi' Y_{lm}^*(\theta', \phi') \tag{3.60}$$

From

$$Y_{2,1}(\theta', \phi') = -\frac{1}{2}\sqrt{\frac{15}{2\pi}} \sin\theta' \cos\theta' e^{i\phi'}, \quad Y_{2,-1}(\theta', \phi') = \frac{1}{2}\sqrt{\frac{15}{2\pi}} \sin\theta' \cos\theta' e^{-i\phi'} \tag{3.61}$$

we get

$$\sin\theta' \cos\theta' \cos\phi' = \sin\theta' \cos\theta' \frac{e^{i\phi'} + e^{-i\phi'}}{2} = -\sqrt{\frac{2\pi}{15}} \left(Y'_{2,1} - Y'_{2,-1} \right) \tag{3.62}$$

where the primes denote the primed variables rather than derivative. Then

$$\int_S d\Omega' \frac{\sin\theta' \cos\theta' \cos\phi'}{|\mathbf{r} - \mathbf{r}'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta, \phi) \int_S d\Omega' \sin\theta' \cos\theta' \cos\phi' Y_{lm}^*(\theta', \phi') \tag{3.63}$$

$$= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta, \phi) \sqrt{\frac{2\pi}{15}} \left(\delta_{m,-1} - \delta_{m,1} \right) \delta_{l,2} \tag{3.64}$$

$$= \frac{4\pi}{5} \frac{r_<^2}{r_>^3} \sqrt{\frac{2\pi}{15}} \left(Y_{2,-1}(\theta, \phi) - Y_{2,1}(\theta, \phi) \right) \tag{3.65}$$

$$= \frac{4\pi}{5} \frac{a^2}{r^3} \sin\theta \cos\theta \cos\phi \tag{3.66}$$

and

$$\Phi_M = \frac{a^2}{4\pi} 3\omega a \epsilon_0 E_0 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \int_S d\Omega' \frac{\sin\theta' \cos\theta' \cos\phi'}{|\mathbf{r} - \mathbf{r}'|} = \frac{a^2}{4\pi} 3\omega a \epsilon_0 E_0 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \frac{4\pi}{5} \frac{a^2}{r^3} \sin\theta \cos\theta \cos\phi \tag{3.67}$$

$$= \frac{3\omega \epsilon_0 E_0}{5} \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \frac{a^5}{r^3} \sin\theta \cos\theta \cos\phi \tag{3.68}$$

$$= \boxed{\frac{3\omega \epsilon_0 E_0}{5} \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \frac{a^5}{r^5} xz} \tag{3.69}$$

The magnetic field outside

$$\mathbf{H} = -\nabla \Phi_M = \boxed{\frac{3\omega \epsilon_0 E_0}{5} \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \left[\frac{(5x^2 - r^2)z}{r^7} \hat{\mathbf{x}} + \frac{5xyz}{r^7} \hat{\mathbf{y}} + \frac{(5z^2 - r^2)x}{r^7} \hat{\mathbf{z}} \right]} \tag{3.70}$$

Uniformly magnetized sphere Compute $\mathbf{H}$, $\mathbf{B}$ and the vector potential $\mathbf{A}$ of a sphere with $\mathbf{M} = M_0\hat{\mathbf{z}}$ and radius a. Since there are no induction currents we may use the magnetic scalar potential Φ_M. Note the effective magnetic charge density $\rho_M = -\nabla\cdot\mathbf{M} = 0$ and $\sigma_M = \hat{\mathbf{n}}\cdot\mathbf{M} = M_0\hat{\mathbf{r}}\cdot\hat{\mathbf{z}} = M_0\cos\theta$, the scalar potential

$$\Phi_M(\mathbf{r}) = \frac{1}{4\pi}\int_V d^3r'\frac{-\nabla'\cdot\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \frac{1}{4\pi}\oint_{\partial V} d^2A'\frac{\hat{\mathbf{n}}'\cdot\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{3.71}$$

$$= \frac{1}{4\pi}\int_0^\pi a^2 d\theta'\sin\theta'\int_0^{2\pi}d\phi'\frac{M_0\cos\theta'}{|\mathbf{r}-\mathbf{r}'|} \tag{3.72}$$

$$= \frac{M_0a^2}{4\pi}\int d\Omega'\cos\theta'\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')\,Y_{lm}(\theta,\phi) \tag{3.73}$$

$$= \frac{M_0a^2}{4\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}Y_{lm}(\theta,\phi)\frac{r_<^l}{r_>^{l+1}}2\sqrt{\frac{\pi}{3}}\int d\Omega' Y_{1,0}(\theta',\phi')\,Y_{lm}^*(\theta',\phi') \tag{3.74}$$

$$= \frac{M_0a^2}{4\pi}\frac{4\pi}{3}2\sqrt{\frac{\pi}{3}}Y_{1,0}(\theta,\phi)\frac{r_<}{r_>^2} \tag{3.75}$$

$$= \frac{M_0a^2}{3}\frac{r_<}{r_>^2}\cos\theta \tag{3.76}$$

So

$$\Phi_M(\mathbf{r}) = \begin{cases}\frac{M_0}{3}r\cos\theta & r<a\\ \frac{M_0}{3}\frac{a^3}{r^2}\cos\theta & r>a\end{cases} \tag{3.77}$$

and the magnetic field inside

$$\mathbf{H}_{\text{in}} = -\nabla\Phi_M = -\nabla\left(\frac{M_0}{3}z\right) = -\frac{\mathbf{M}}{3} \tag{3.78}$$

with the constitutive equation $\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M})$ we have

$$\mathbf{B}_{\text{in}} = \frac{2}{3}\mu_0\mathbf{M} \tag{3.79}$$

To conclude

$$\boxed{\mathbf{H}_{\text{in}} = -\frac{\mathbf{M}}{3},\quad \mathbf{B}_{\text{in}} = \frac{2}{3}\mu_0\mathbf{M}} \tag{3.80}$$

The vector potential is induced by the effective magnetic surface current density $\mathbf{j}_M = -\hat{\mathbf{n}}\times\mathbf{M}$ and the magnetization current $\mathbf{J}_M = \nabla\times\mathbf{M}$. The former

$$\mathbf{j}_M = -\hat{\mathbf{n}}\times\mathbf{M} = -M_0\hat{\mathbf{r}}\times\hat{\mathbf{z}} = M_0\sin\theta\hat{\phi} \tag{3.81}$$

and the latter $\mathbf{J}_M = \nabla\times\mathbf{M} = 0$ because $\mathbf{M}_{\text{in}}$ is uniform. Then

$$\mathbf{A} = \frac{\mu_0}{4\pi}\oint dA'\frac{\mathbf{j}_M(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mu_0M_0}{4\pi}\oint dA'\frac{\sin\theta'\hat{\phi}'}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mu_0M_0}{4\pi}\int d\Omega'\frac{\sin\theta'(-\sin\phi'\hat{\mathbf{x}}+\cos\phi'\hat{\mathbf{y}})}{|\mathbf{r}-\mathbf{r}'|} \tag{3.82}$$

The x-component

$$A_x = -\frac{\mu_0M_0}{4\pi}\int d\Omega'\frac{\sin\theta'\sin\phi'}{|\mathbf{r}-\mathbf{r}'|} = -\frac{\mu_0M_0}{4\pi}\int d\Omega'\sin\theta'\sin\phi'\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')\,Y_{lm}(\theta,\phi) \tag{3.83}$$

$$= -\frac{\mu_0M_0a^2}{4\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\phi)\frac{1}{i}\sqrt{\frac{3}{2\pi}}\int d\Omega'\left(Y_{1,1}'+Y_{1,-1}'\right)Y_{lm}'^{\,*} \tag{3.84}$$

$$= -\frac{\mu_0M_0a^2}{4\pi}\frac{4\pi}{3}\frac{r_<}{r_>^2}\frac{1}{i}\sqrt{\frac{3}{2\pi}}\left[Y_{1,1}(\theta,\phi)+Y_{1,-1}(\theta,\phi)\right] \tag{3.85}$$

$$= -\frac{\mu_0M_0a^2}{3}\frac{r_<}{r_>^2}\sin\theta\sin\phi \tag{3.86}$$

and similarly

$$A_y = \frac{\mu_0 M_0 a^2}{4\pi}\int d\Omega' \frac{\sin\theta'\cos\phi'}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mu_0 M_0}{3}\frac{r_<}{r_>^2}\sin\theta\cos\phi \tag{3.87}$$

giving

$$\mathbf{A} = \frac{\mu_0 M_0 a^2}{3}\frac{r_<}{r_>^2}\sin\theta\left(-\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}\right) = \boxed{\frac{\mu_0 M_0 a^2}{3}\frac{r_<}{r_>^2}\sin\theta\hat{\phi}} \tag{3.88}$$

Uniformly magnetized sphere in an otherwise constant B Consider a region of space containing a constant magnetic induction $\mathbf{B} = B_0\hat{\mathbf{z}}$. Find the magnetization $\mathbf{M}$ inside the sphere.

Suppose the magnetization inside the sphere is $\mathbf{M}$. Given that the field equations are linear, the magnetic induction inside the sphere is the superposition of the induction purely due to magnetization and the external magnetic induction

$$\mathbf{B}_{\text{in}} = \frac{2}{3}\mu_0\mathbf{M} + \mathbf{B} \tag{3.89}$$

Similarly the magnetic field inside

$$\mathbf{H}_{\text{in}} = -\frac{\mathbf{M}}{3} + \frac{\mathbf{B}}{\mu_0} \tag{3.90}$$

Now the constitutive relation $\mathbf{B}_{\text{in}} = \mu\mathbf{H}_{\text{in}}$ gives

$$\frac{2}{3}\mu_0\mathbf{M} + \mathbf{B} = \mu\left(-\frac{\mathbf{M}}{3} + \frac{\mathbf{B}}{\mu_0}\right) \tag{3.91}$$

then

$$\boxed{\mathbf{M} = \frac{3}{\mu_0}\frac{\mu-\mu_0}{\mu+2\mu_0}\mathbf{B}} \tag{3.92}$$

Permeable sphere in an otherwise constant B Consider a region of space containing a constant magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$

4 Quasi-static fields

4.1 Faraday's law

Faraday's law relates time-dependent electric and magnetic fields. It states that the magnetic flux passing an area S, $\mathcal{F}_B \equiv \int_S d^2A\hat{\mathbf{n}}\cdot\mathbf{B}$, induces an electromotive force (emf) $\mathcal{E} = \oint_{\partial S} d\mathbf{l}\cdot\mathbf{E}$ via

$$\mathcal{E} = -\frac{d\mathcal{F}_B}{dt}, \quad \oint_{\partial S} d\mathbf{l}\cdot\mathbf{E} = -\frac{d}{dt}\int_S d^2A\hat{\mathbf{n}}\cdot\mathbf{B} \tag{4.1}$$

Using the Stokes theorem we obtain the differential form of Faraday's law

$$\boxed{\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}} \tag{4.2}$$

4.2 Quasi-static fields in uniform media

The regime where the time variation in electromagnetic fields are not too rapid, such that the finite speed of light can be approximated as infinite and fields can be treated as they propagate instantaneously, is called quasi-static. In the following we investigate the diffusion of fields in the quasi-static regime.

Diffusion equation For a uniform media with uniform field-independent permeability μ and conductivity σ, the fields inside obey the equations

$$\nabla \times \mathbf{H} = \mathbf{J}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \mathbf{J} = \sigma \mathbf{E} \tag{4.3}$$

where the last one is Ohm's law. With the constitutive equation $\mathbf{B} = \mu \mathbf{H}$ we take a curl of the first equation

$$\nabla \times (\nabla \times \mathbf{H}) = \nabla \times \mathbf{J} \tag{4.4}$$

$$\nabla (\nabla \cdot \mathbf{H}) - \nabla^2 \mathbf{H} = \sigma \nabla \times \mathbf{E} = -\mu\sigma \frac{\partial \mathbf{H}}{\partial t} \tag{4.5}$$

This leads to the diffusion equations for the magnetic field and induction

$$\boxed{\nabla^2 \mathbf{H} = \mu\sigma \frac{\partial \mathbf{H}}{\partial t}, \quad \nabla^2 \mathbf{B} = \mu\sigma \frac{\partial \mathbf{B}}{\partial t}} \tag{4.6}$$

Since $\mathbf{B} = \nabla \times \mathbf{A}$, taking the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ yields the diffusion equation for the vector potential

$$\boxed{\nabla^2 \mathbf{A} = \mu\sigma \frac{\partial \mathbf{A}}{\partial t}} \tag{4.7}$$

Skin depth (JDJ sec. 5.18A) A semi-infinite conductor of uniform conductivity σ and permeability μ occupies the space $z > 0$, with vacuum for $z < 0$. At the surface $z = 0^-$ a time-dependent but spatially uniform magnetic field is present, $H_x(t, z = 0^-) = H_0 e^{i\omega t}$. We seek for a steady-state solution inside this conductor, subject to the boundary conditions at $z = 0^+$and $z \to \infty$.

At the surface $z = 0^-$, since there's no conducting currents, the tangential component of $\mathbf{H}$ is continuous, so $H_x(t, z = 0^+) = H_x(t, z = 0^-) = H_0 e^{i\omega t}$. Try the ansatz for steady-state solution

$$H_x(t, z) = h(z) e^{-i\omega t} \tag{4.8}$$

where, from Eq. (4.6), $h(z)$ satisfies

$$\frac{d^2 h(z)}{dz^2} + i\mu\sigma\omega h(z) = 0 \tag{4.9}$$

which has solution

$$h(z) = c_1 e^{ikz} + c_2 e^{ikz}; \quad k^2 = i\mu\sigma\omega \quad \text{or} \quad k = \sqrt{\frac{\mu\sigma\omega}{2}}(1+i) \tag{4.10}$$

Call

$$\delta \equiv \sqrt{\frac{2}{\mu\sigma\omega}} \tag{4.11}$$

the skin depth, we find

$$H_x(t, z) = c_1 e^{-z/\delta} e^{i(z/\delta - \omega t)} + c_2 e^{z/\delta} e^{i(-z/\delta - \omega t)} \tag{4.12}$$

The coefficient $c_1 = H_0$ as $H_x(t, z = 0^-) = H_0 e^{i\omega t}$ and $c_2 = 0$ since $H_x(t, z)$ has to vanish as $z \to \infty$, giving the steady-state solution in the media

$$\boxed{H_x(t, z) = H_0 e^{-z/\delta} e^{i(z/\delta - \omega t)}, \quad z > 0} \tag{4.13}$$

Diffusion of magnetic fields in conducting media (JDJ sec. 5.18B) Two infinite current sheets, parallel to each other and located at $z=-a$ and $z=+a$, within an infinite conducting medium of permeability μ and conductivity σ. The currents are such that in the region $\{0 \le |z| < a\}$ there is a constant field $H_0\hat{\mathbf{x}}$ and zero field outside. At $t=0$ the current is suddenly turned off. We seek for the time-evolution of the magnet field by solving the diffusion equation.

Taking a Laplace transform of $H_x(t,z)$

$$H_x(t,z) = \int_0^\infty ds e^{-st} h_x(s,z) \tag{4.14}$$

then a Fourier transform of $h(s,z)$

$$h_x(s,z) = \int_{-\infty}^\infty \frac{dk}{2\pi} e^{ikz} \tilde{h}_x(s,k) \tag{4.15}$$

we get

$$H_x(t,z) = \int_0^\infty ds e^{-st} \int_{-\infty}^\infty \frac{dk}{2\pi} e^{ikz} \tilde{h}_x(s,k) \tag{4.16}$$

The diffusion equation of $\tilde{h}_x(s,k)$ reads

$$\int_0^\infty ds e^{-st} \int_{-\infty}^\infty \frac{dk}{2\pi} \left(k^2 - \mu\sigma s\right) \tilde{h}_x(s,k) = 0 \tag{4.17}$$

so $k^2 = \mu\sigma s$ and

$$H_x(t,z) = \int_{-\infty}^\infty \frac{dk}{2\pi} e^{-\frac{k^2 t}{\mu\sigma}} e^{ikz} \tilde{h}(k) \tag{4.18}$$

where $\tilde{h}(k) \equiv \tilde{h}_x(s,k)$ is a shorthand for the Fourier coefficients.

From the initial condition

$$H_x\left(t=0^+, z\right) = \int_{-\infty}^\infty \frac{dk}{2\pi} e^{ikz} \tilde{h}(k) = H_0 \left[\Theta(z+a) - \Theta(z-a)\right] \tag{4.19}$$

inverse Fourier transform leads to

$$\tilde{h}(k) = \int_{-\infty}^\infty dz e^{-ikz} H_0 \left[\Theta(z+a) - \Theta(z-a)\right] = H_0 \int_{-a}^a dz e^{-ikz} = \frac{2H_0}{k}\sin(ka) \tag{4.20}$$

So

$$H_x\left(t=0^+, z\right) = \int_{-\infty}^\infty \frac{dk}{2\pi} e^{-\frac{k^2 t}{\mu\sigma}} e^{ikz} \tilde{h}(k) = \frac{H_0}{\pi} \int_{-\infty}^\infty dk e^{-\frac{k^2 t}{\mu\sigma}} e^{ikz} \frac{\sin(ka)}{k} \tag{4.21}$$

$$= \frac{H_0}{\pi} \left(\int_0^\infty dk e^{ikz} e^{-\frac{k^2 t}{\mu\sigma}} \frac{\sin(ka)}{k} + \int_{-\infty}^0 dk e^{ikz} e^{-\frac{k^2 t}{\mu\sigma}} \frac{\sin(ka)}{k} \right) \tag{4.22}$$

$$= \frac{2H_0}{\pi} \int_0^\infty dk \left(\frac{e^{ikz} + e^{-ikz}}{2} \right) e^{-\frac{k^2 t}{\mu\sigma}} \frac{\sin(ka)}{k} \tag{4.23}$$

$$= \frac{2H_0}{\pi} \int_0^\infty dk e^{-\frac{k^2 t}{\mu\sigma}} \frac{\cos(kz)\sin(ka)}{k} \tag{4.24}$$

$$= H_0 \frac{2}{\pi} \int_0^\infty d\kappa e^{-\nu t \kappa^2} \frac{\cos(\kappa z/a)\sin(\kappa)}{\kappa} \tag{4.25}$$

where $\nu \equiv 1/\left(\mu\sigma a^2\right)$ is a characteristic decay rate. This integral can be written in terms of the error function

$$\Phi(\xi) \equiv \frac{2}{\sqrt{\pi}} \int_0^\xi dx e^{-x^2} \equiv \frac{2}{\pi} \int_0^\infty dx e^{-\frac{x^2}{4\xi^2}} \frac{\sin x}{x} \tag{4.26}$$

We note

$$\cos(\kappa z/a)\sin(\kappa) = \frac{1}{2}\left[\sin(\kappa(1+z/a)) + \sin(\kappa(1-z/a))\right] \tag{4.27}$$

then

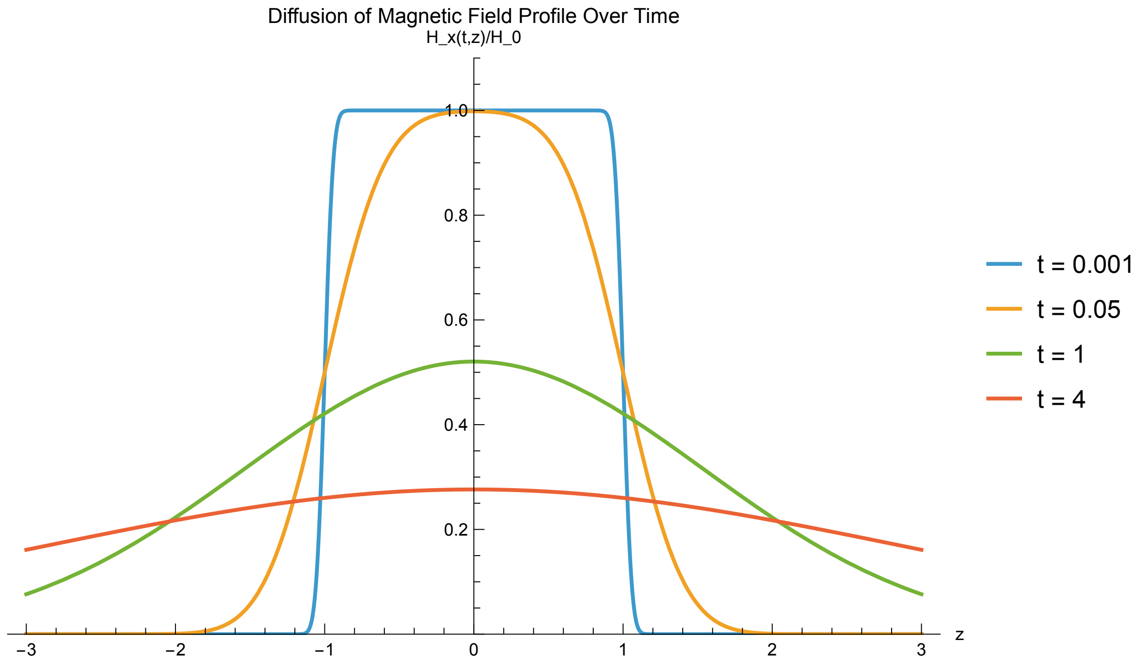
$$\frac{2}{\pi}\int_0^\infty d\kappa e^{-\nu t\kappa^2}\frac{\cos(\kappa z/a)\sin(\kappa)}{\kappa} = \frac{1}{2}\frac{2}{\pi}\int_0^\infty d\kappa e^{-\nu t\kappa^2}\left(\frac{\sin(\kappa(1+z/a)) + \sin(\kappa(1-z/a))}{\kappa}\right) \tag{4.28}$$

$$= \frac{1}{2}\left[\frac{2}{\pi}\int_0^\infty d\kappa' e^{-\frac{\nu t\kappa'^2}{(1+z/a)^2}}\frac{\sin(\kappa')}{\kappa'} + \frac{2}{\pi}\int_0^\infty d\kappa' e^{-\frac{\nu t\kappa'^2}{(1-z/a)^2}}\frac{\sin(\kappa')}{\kappa'}\right] \tag{4.29}$$

$$= \frac{1}{2}\left[\Phi\left(\frac{1+z/a}{2\sqrt{\nu t}}\right) + \Phi\left(\frac{1-z/a}{2\sqrt{\nu t}}\right)\right] \tag{4.30}$$

The final result is

$$\boxed{H_x\left(t=0^+, z\right) = \frac{H_0}{2}\left[\Phi\left(\frac{1+z/a}{2\sqrt{\nu t}}\right) + \Phi\left(\frac{1-z/a}{2\sqrt{\nu t}}\right)\right]} \tag{4.31}$$



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