PHY322 S26
Discussion 1
TA: Junqi Zhang

Overview

The objective of PHY322 discussions is twofolds.

1. Assist you work out homework problems by pointing out the right direction. Note the homework is due after the discussion date.
2. Supplement the lectures with details.

General policies for the discussion sections:

- TA office hours: Mondays 2:30 - 4:30PM. Location: Chamberlin Hall 2307.
- Reach me by email at jzhang2648@wisc.edu
- Attendance is NOT mandatory, but keep in mind that the discussions happen before turning in your homework.
- Please (at least) read the homework problems before the discussions.
- Interrupt me whenever you have a question.
- Printed handouts will be distributed before the end of the discussions.
- The supplement materials in the handouts will likely not be covered in class. These are mostly from my notes and will be served as desserts if you have appetite *after* digesting the lectures.
- Any feedback is welcome.

Miscellaneous stuff to mention:

- Use AI wisely. Nowadays AI can easily solve most of your homework problems with a high accuracy. However, if your goal in this course is to gain a better understanding of electrodynamics, it is better to utilize AI as a smart search engine and an assistant rather than an answer generator. For example, you may use AI to
  - Convert your handwritings into LaTeX
  - Help you write code to check your calculations
  - Collect information in an organized way
  - ..., you name it

Let's go.

Homework 1

Problems 1&2

We introduce two gadgets for solving problems 1 and 2:

- Einstein summation convention
- Levi-Civita symbol

They provide a powerful language that not only prevents notational clutter, but also helps you see through the structure of index manipulations. In the following we will first introduce the Einstein summation conventions with examples, then introduce the Levi-Civita symbol and derive a useful identity. Problems 1 and 2 are in principle solvable by brutal force computations, but with these gadgets you will be able to solve a wider range of vector calculation problems systematically.

Einstein summation conventions (for problem 2 and Levi-Civita symbol)

The vector $\vec{A} = (A_x, A_y, A_z) \in \mathbb{R}^3$ can be written in the component form

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}\,. \tag{1}$$

Alternatively, we introduce the notations

$$\begin{cases} \hat{e}_1 \equiv \hat{x} \\ \hat{e}_2 \equiv \hat{y} \\ \hat{e}_3 \equiv \hat{z} \end{cases} , \quad \begin{cases} A_1 \equiv A_x \\ A_2 \equiv A_y \\ A_3 \equiv A_z \end{cases} \tag{2}$$

then the component form (1) translates to[1]

$$\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3 = \sum_{i=1}^{3} A_i\hat{e}_i\,. \tag{3}$$

In the Einstein summation conventions, the above is written as

$$\vec{A} = \sum_{i=1}^{3} A_i\hat{e}_i \to A_i\hat{e}_i\,, \tag{4}$$

where the summation symbol is omitted for brevity. The general rule in the Einstein summation conventions is that: *whenever an abstract index appears twice, the object is summed over all values of that index*. For example,

- the dot product of $\vec{A} = (A_x, A_y, A_z)$ and $\vec{B} = (B_x, B_y, B_z)$ can be written as

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z = \sum_{i=1}^{3} A_iB_i \to A_iB_i \tag{5}$$

  where $B_1 \equiv B_x,\ B_2 \equiv B_y$ and $B_3 \equiv B_z$.

- The divergence $\vec{\nabla}\cdot\vec{A}$ can be written as

$$\vec{\nabla}\cdot\vec{A} = \partial_x A_x + \partial_y A_y + \partial_z A_z \to \partial_i A_i \tag{6}$$

  where $\partial_1 \equiv \frac{\partial}{\partial x},\ \partial_2 \equiv \frac{\partial}{\partial y}$ and $\partial_3 \equiv \frac{\partial}{\partial z}$.

Try to use the Einstein summation conventions to solve **problem 2**.

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[1] Physicists usually use "$\equiv$" as a defining operator. For instance, $a \equiv b$ means that a is defined as b.

Levi-Civita symbol (for problem 1)

The motivation for introducing the Levi-Civita symbol can be seen from the cross product of 3d vectors. Note

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}=\left(A_yB_z-A_zB_y\right)\hat{x}+\left(A_zB_x-A_xB_z\right)\hat{y}+\left(A_xB_y-A_yB_x\right)\hat{z} \tag{7}$$

we have

$$\left(\vec{A}\times\vec{B}\right)_1 = A_2B_3-A_3B_2 \tag{8}$$

$$\left(\vec{A}\times\vec{B}\right)_2 = A_3B_1-A_1B_3 \tag{9}$$

$$\left(\vec{A}\times\vec{B}\right)_3 = A_1B_2-A_2B_1 \tag{10}$$

To rewrite the above expressions in a compact way, we introduce the Levi-Civita symbol defined as

$$\epsilon_{ijk}\equiv\begin{cases}1 & (i,j,k)\in\{(1,2,3),(3,1,2),(2,3,1)\}\\ -1 & (i,j,k)\in\{(1,3,2),(3,2,1),(2,1,3)\}\\ 0 & \text{otherwise}\end{cases} \tag{11}$$

In the Einstein summation conventions, we can write the i-th component of the vector product $\vec{A}\times\vec{B}$ as

$$\left(\vec{A}\times\vec{B}\right)_i \equiv \epsilon_{ijk}A_jB_k \tag{12}$$

You can check this by explicit calculations. For example,

$$\begin{aligned}\left(\vec{A}\times\vec{B}\right)_1 &= \sum_{j,k\in\{1,2,3\}}\epsilon_{1jk}A_jB_k && (13)\\ &= \epsilon_{111}A_1B_1+\epsilon_{112}A_1B_2+\epsilon_{113}A_1B_3 && (14)\\ &\quad+\epsilon_{121}A_2B_1+\epsilon_{122}A_2B_2+\underbrace{\epsilon_{123}A_2B_3} && (15)\\ &\quad+\epsilon_{131}A_3B_1+\underbrace{\epsilon_{132}A_3B_2}+\epsilon_{133}A_3B_3 && (16)\\ &= \epsilon_{123}A_2B_3+\epsilon_{132}A_3B_2 && (17)\\ &= A_2B_3-A_3B_2 && (18)\end{aligned}$$

With the Levi-Civita symbol, you can rewrite other things such as the curl

$$\vec{\nabla}\times\vec{A}=\left(\frac{\partial}{\partial y}A_z-\frac{\partial}{\partial z}A_y\right)\hat{x}+\left(\frac{\partial}{\partial z}A_x-\frac{\partial}{\partial x}A_z\right)\hat{y}+\left(\frac{\partial}{\partial x}A_y-\frac{\partial}{\partial y}A_x\right)\hat{z} \tag{19}$$

as

$$\left(\vec{\nabla}\times\vec{A}\right)_i=\epsilon_{ijk}\partial_jA_k \tag{20}$$

Note that the Levi-Civita symbol is anti-symmetric in its indices. Specifically, in ϵ_{ijk}, swapping any two indices yields a minus sign

$$\epsilon_{ijk}=-\epsilon_{jik}\ (\text{swap } i,j) \tag{21}$$

$$\epsilon_{ijk}=-\epsilon_{ikj}\ (\text{swap } j,k) \tag{22}$$

$$\epsilon_{ijk}=-\epsilon_{kji}\ (\text{swap } i,k) \tag{23}$$

With this anti-symmetry, we can show an important identity that can help you solve **problem 1**

$$\boxed{\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}} \tag{24}$$

where i, j, l and m are given indices and we have introduced the Kronecker delta symbol

$$\delta_{ij} \equiv \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{25}$$

To show eq. (24), we rewrite it in the complete form

$$\epsilon_{ijk}\epsilon_{lmk} \to \sum_{k=1}^{3} \epsilon_{ijk}\epsilon_{lmk} = \epsilon_{ij1}\epsilon_{lm1} + \epsilon_{ij2}\epsilon_{lm2} + \epsilon_{ij3}\epsilon_{lm3} \tag{26}$$

From the anti-symmetry property of ϵ_{ijk}, we know that if any two of the indices $\{i, j, k\}$ are equal, then $\epsilon_{ijk} = 0$. This means that for each given k value in the summation $\sum_{k=1}^{3}$, the indices of nonzero ϵ_{ijk} must satisfy

$$i \neq k \quad \text{and} \quad j \neq k \tag{27}$$

or equivalently,

$$\{i, j | \epsilon_{ijk} \neq 0\} = \{1, 2, 3\} \setminus \{k\} \tag{28}$$

Similarly, the anti-symmetry of ϵ_{lmk} implies that the indices of nonzero ϵ_{lmk} satisfy

$$\{l, m | \epsilon_{lmk} \neq 0\} = \{1, 2, 3\} \setminus \{k\} \tag{29}$$

Therefore

$$\{i, j | \epsilon_{ijk} \neq 0\} = \{l, m | \epsilon_{lmk} \neq 0\} = \{1, 2, 3\} \setminus \{k\} \tag{30}$$

Furthermore, for a given k, the indices of nonzero $\epsilon_{ijk}\epsilon_{lmk}$ have to satisfy $i \neq j$ and $l \neq m$, which means

$$\text{either} \quad \{i = l,\ j = m\} \quad \text{or} \quad \{i = m,\ j = l\}\ . \tag{31}$$

For example, if we focus on the $k = 1$ term in the summation (26), then the indices of nonzero $\epsilon_{ij1}\epsilon_{lm1}$ satisfy

$$\{i, j | \epsilon_{ij1} \neq 0\} = \{l, m | \epsilon_{lm1} \neq 0\} = \{2, 3\} \quad \text{and} \quad \{i \neq j,\ l \neq m\} \tag{32}$$

which means the nonzero terms are

$$\epsilon_{231}\epsilon_{231} \quad \text{and} \quad \epsilon_{321}\epsilon_{321}\ . \tag{33}$$

Hence, according to eq. (31), we conclude that

$$\epsilon_{ijk}\epsilon_{lmk} = c_1\delta_{il}\delta_{jm} + c_2\delta_{im}\delta_{jl} \tag{34}$$

with c_1 and c_2 two constants and Kronecker deltas that enforce the condition (31). Now we determine the constants. In eq. (34), if $i = l = 1$ and $j = m = 2$, then on one hand

$$\epsilon_{12k}\epsilon_{12k} = c_1\delta_{11}\delta_{22} + c_2\delta_{12}\delta_{21} = c_1 \tag{35}$$

and on the other hand

$$\epsilon_{12k}\epsilon_{12k} \to \sum_{k=1}^{3} \epsilon_{12k}\epsilon_{12k} = \epsilon_{123}\epsilon_{123} = 1 \tag{36}$$

which gives $c_1 = 1$. Similarly, in eq. (34) taking $i = m = 1$ and $j = l = 2$ gives

$$\epsilon_{12k}\epsilon_{21k} = c_2 = -\epsilon_{12k}\epsilon_{12k} = -c_1 = -1 \tag{37}$$

As a result, eq. (34) becomes

$$\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\ . \tag{38}$$

Problems 3-5

In these problems you are asked to verify the divergence theorem and the Stokes theorem via explicit calculations. In general, this type of theorems establishes a relation between bulk and boundary, which could be summarized as

$$\int_{\text{bulk}} (d\,\text{volume})\,(\text{differential of vector}) = \int_{\text{boundary}} (d\,\text{surface})\,(\text{vector}) \tag{39}$$

For example, the divergence theorem reads

$$\int_V d\tau \left(\vec{\nabla} \cdot \vec{A}\right) = \oint_{\partial V} d\vec{a} \cdot \vec{A} \tag{40}$$

in which we can find the correspondences

$$\text{bulk} \to V, \quad \text{boundary} \to \partial V, \tag{41}$$

$$(d\,\text{volume}) \to d\tau, \quad (d\,\text{surface}) \to d\vec{a} \tag{42}$$

$$(\text{differential of vector}) \to \vec{\nabla} \cdot \vec{A}, \quad (\text{vector}) \to \vec{A} \tag{43}$$

The Stokes theorem reads

$$\int_S d\vec{a} \cdot \left(\vec{\nabla} \times \vec{A}\right) = \oint_{\partial S} d\vec{l} \cdot \vec{A} \tag{44}$$

in which

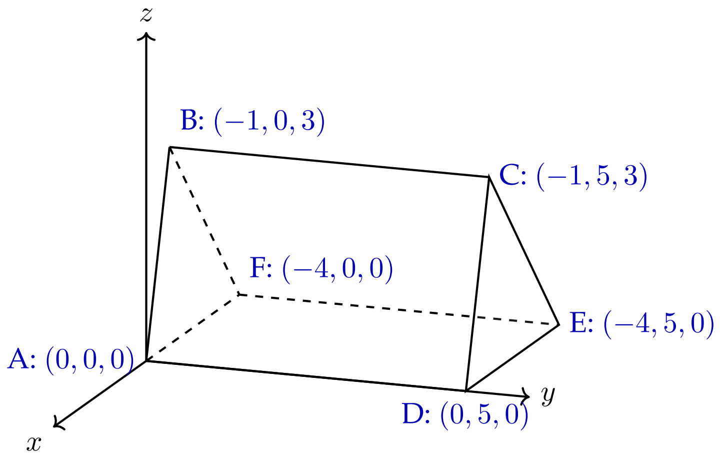
$$\text{bulk} \to S, \quad \text{boundary} \to \partial S, \tag{45}$$

$$(d\,\text{volume}) \to d\vec{a}, \quad (d\,\text{surface}) \to d\vec{l} \tag{46}$$

$$(\text{differential of vector}) \to \vec{\nabla} \times \vec{A}, \quad (\text{vector}) \to \vec{A} \tag{47}$$

After this conceptualization, we consider technical aspects of performing these integrals. In these problems, one of the technical obstacles is to calculate $d\vec{a}$ or $d\vec{l}$. For $d\vec{a}$, we will first draw a diagram to understand the geometry, then express $d\vec{a}$ as a cross product of infinitesimal line elements on the surface. For $d\vec{l}$, what we will do is to parametrize the curve and take differential.

Take **problem 3** for example. You are asked to perform surface integral of the vector field $\vec{A} = x\hat{x} + y\hat{y} + z^2\hat{z}$ on a polyhedron given by



We calculate the integral

$$\int_{\text{Face ADCB}} d\vec{a}\cdot\vec{A} \tag{48}$$

on the face ADCB as a prototype. To do this, we need to know what are the values of the normal vector $d\vec{a}$ and $\vec{A}$ restricted on this face. As a first step, we write down the equation for the face ADCB. The normal vector of ADCB is proportional to $\overrightarrow{AD}\times\overrightarrow{AB} = 15\hat{x}+5\hat{z}$, so after normalization

$$\hat{n}_{\text{ADCB}} = \frac{1}{\sqrt{10}}\left(3\hat{x}+\hat{z}\right) \tag{49}$$

For any point $p=(x,y,z)\in$ ADCB we have $(x,y,z)\cdot\hat{n}_{\text{ADCB}}=0$, giving

$$3x+z=0 \tag{50}$$

which then gives a constraint on the differentials

$$dz=-3dx \tag{51}$$

At p, the normal vector $d\vec{a}$ should be proportional to $\hat{n}_{\text{ADCB}}$, which can be seen from geometry [2]

$$d\vec{a} = dy\hat{y}\times(-dx\hat{x}+dz\hat{z}) = dxdy\hat{z}+dydz\hat{x} \tag{52}$$

With the constraints (50) and (51) we compute

$$\begin{aligned}
\int_{\text{Face ADCB}} d\vec{a}\cdot\vec{A} &= \int_{\text{Face ADCB}} \left(dxdy\hat{z}+dydz\hat{x}\right)|_{dz=-3dx}\cdot\left(x\hat{x}+y\hat{y}+z^2\hat{z}\right)|_{z=-3x} && (53)\\
&= \int_{\text{Face ADCB}} \left[dxdyz^2+dydzx\right]_{z=-3x,\ dz=-3dx} && (54)\\
&= \cdots && (55)
\end{aligned}$$

Now consider how to compute $d\vec{l}$ in **problem 4**. Note the circle $C=\{(x,y)|x^2+y^2=1\}$ has the parametrization

$$x=\cos s,\quad y=\sin s,\qquad s\in[0,2\pi] \tag{56}$$

which gives

$$d\vec{l} = dx\hat{x}+dy\hat{y} = ds\left(\frac{dx}{ds}\hat{x}+\frac{dy}{ds}\hat{y}\right) = ds\left(-\sin s\hat{x}+\cos s\hat{y}\right) \tag{57}$$

Consequently, the line integral

$$\begin{aligned}
I &\equiv \oint_C d\vec{l}\cdot\left(y\hat{x}+2x\hat{y}+z^2\hat{z}\right) && (58)\\
&= \oint_C ds\left(-\sin s\hat{x}+\cos s\hat{y}\right)\cdot\left(y\hat{x}+2x\hat{y}+z^2\hat{z}\right)_{x=\cos s,\ y=\sin s} && (59)\\
&= \int_0^{2\pi} ds\left(-y\sin s+2x\cos s\right)_{x=\cos s,\ y=\sin s} && (60)\\
&= \cdots && (61)
\end{aligned}$$

You can use this approach to calculate other integrals in problems 3-5.

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[2] A systematic treatment of $d\vec{a}$ would need us to invoke differential forms. If you are interested, find 'differential form' in Wikipedia and ask AI how to apply it to 3d vector analyses, or ask me during my OH.

Lectures 1&2 supplement material

In this supplement material, we give details about the Dirac delta function representations. The conclusions are important but what's more interesting is the technique used in deriving them. You will encounter the contour trick used in the first representation again and again in various Green functions in the context of, for example, electromagnetic radiations. The second representation is arguably the most useful one for formal derivations in periodic systems, such as EM waves, as it tells you how to Fourier transform a constant function. Physically this would correspond to spectral resolution of plane waves. As a bonus, in the derivation for all the these representations you will see how physicists recklessly interchange the order of limit and integration to get a reasonable result, which usually takes mathematicians several textbooks to get there :)

Dirac function representation details

Representation 1 One way to represent the Dirac delta function is

$$\boxed{\delta(x-x_0) \equiv \lim_{\epsilon\to 0}\frac{1}{\pi}\frac{\epsilon}{(x-x_0)^2+\epsilon^2}} \tag{62}$$

We will show that for functions $f(x)$ satisfying certain conditions, the above representation manifests the two properties of the Dirac delta function

1. Normalization $\int_{-\infty}^{\infty} dx\delta(x-x_0) = 1$
2. Pick out points $\int_{-\infty}^{\infty} dx\delta(x-x_0)f(x) = f(x_0)$

To show the normalization condition, we compute

$$\int_{-\infty}^{\infty} dx\delta(x-x_0) = \int_{-\infty}^{\infty} dx \lim_{\epsilon\to 0}\frac{1}{\pi}\frac{\epsilon}{(x-x_0)^2+\epsilon^2} = \lim_{\epsilon\to 0}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{d\left(\frac{x-x_0}{\epsilon}\right)}{1+\left(\frac{x-x_0}{\epsilon}\right)^2} \tag{63}$$

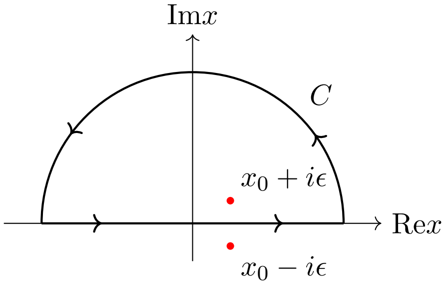
$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dy}{1+y^2} = \frac{1}{\pi}\tan^{-1}(y)\Big|_{-\infty}^{\infty} = 1 \tag{64}$$

The picking-out-points property can be seen as follows

$$\int_{-\infty}^{\infty} dx\delta(x-x_0)f(x) = \int_{-\infty}^{\infty} dx \lim_{\epsilon\to 0}\frac{1}{\pi}\frac{\epsilon}{(x-x_0)^2+\epsilon^2}f(x) \tag{65}$$

$$= \frac{1}{2\pi i}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dx\left(\frac{f(x)}{x-x_0-i\epsilon} - \frac{f(x)}{x-x_0+i\epsilon}\right) \tag{66}$$

If $f(x)$ could be analytically continued to the complex plane, then with Cauchy's theorem we evaluate the integral along the contour shown below



which returns

$$\int_{-\infty}^{\infty} dx\delta(x-x_0)f(x) = \frac{1}{2\pi i}\lim_{\epsilon\to 0}\int_C dx\frac{f(x)}{x-x_0-i\epsilon} = \lim_{\epsilon\to 0} f(x_0+i\epsilon) = f(x_0) \tag{67}$$

Hence the conditions 1 and 2 are satisfied, proving that eq. (62) is a Dirac function representation.

Representation 2 Another useful representation of the Dirac delta function is

$$\boxed{\delta(x-x_0) \equiv \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x_0)}} \tag{68}$$

This integral is seemingly divergent. To make this divergent integral work, we add a convergent parameter as follows

$$\begin{aligned}\delta(x-x_0) &\equiv \int_{-\infty}^{0} \frac{dk}{2\pi} e^{ik(x-x_0)} + \int_{0}^{\infty} \frac{dk}{2\pi} e^{ik(x-x_0)} = \lim_{\epsilon\to 0^+}\left(\int_{-\infty}^{0} \frac{dk}{2\pi} e^{ik(x-x_0-i\epsilon)} + \int_{0}^{\infty} \frac{dk}{2\pi} e^{ik(x-x_0+i\epsilon)}\right) \\ &= \frac{1}{2\pi i}\lim_{\epsilon\to 0^+}\left(\left.\frac{e^{ik(x-x_0-i\epsilon)}}{x-x_0-i\epsilon}\right|_{k=-\infty}^{k=0} + \left.\frac{e^{ik(x-x_0+i\epsilon)}}{x-x_0+i\epsilon}\right|_{k=0}^{k=\infty}\right) = \frac{1}{2\pi i}\lim_{\epsilon\to 0^+}\left(\frac{1}{x-x_0-i\epsilon} - \frac{1}{x-x_0+i\epsilon}\right) \\ &= \lim_{\epsilon\to 0^+}\frac{1}{\pi}\frac{\epsilon}{(x-x_0)^2+\epsilon^2} \end{aligned} \tag{69}$$

which equals to the first representation of the Dirac delta function eq. (62).

Representation 3 The Dirac delta function can also be represented by an improper limit of Gaussian distribution functions

$$\boxed{\delta(x-x_0) \equiv \lim_{\epsilon\to 0}\frac{1}{\sqrt{\pi\epsilon^2}}\exp\left(-\frac{(x-x_0)^2}{\epsilon^2}\right)} \tag{70}$$

We show this by proving the shifting property. For a smooth function $f(x)$, we note

$$\begin{aligned}\int_{-\infty}^{\infty} dx f(x)\lim_{\epsilon\to 0}\frac{1}{\sqrt{\pi\epsilon^2}}\exp\left(-\frac{(x-x_0)^2}{\epsilon^2}\right) &= \int_{-\infty}^{\infty} dx f(x)\lim_{\epsilon\to 0}\frac{1}{\sqrt{\pi\epsilon^2}}\exp\left(-\frac{(x-x_0)^2}{\epsilon^2}\right) && (71)\\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\lim_{\epsilon\to 0^+}\frac{dx}{\epsilon} f(x)\exp\left(-\left(\frac{x-x_0}{\epsilon}\right)^2\right) && (72)\\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\lim_{\epsilon\to 0^+} dy f(x_0+\epsilon y)\exp\left(-y^2\right) && (73)\\ &= \frac{f(x_0)}{\sqrt{\pi}}\int_{-\infty}^{\infty} dy e^{-y^2} && (74)\\ &= f(x_0) && (75)\end{aligned}$$

which implies

$$\int_{-\infty}^{\infty} dx f(x)\left[\lim_{\epsilon\to 0}\frac{1}{\sqrt{\pi\epsilon^2}}\exp\left(-\frac{(x-x_0)^2}{\epsilon^2}\right)\right] = f(x_0) \tag{76}$$

Taking $f(x)=1$ we can show the normalization condition. Hence eq. (70) is indeed a representation of the Dirac function.