PHY322 S26
Discussion 10
TA: Junqi Zhang

In this discussion we will start by reviewing linear medium, Ohm's law, and Faraday's law covered in lectures 18-19, then talk about homework 9.

Review

As a continuation of the general study of magnetic fields in matter, we discussed qualitative explanations of paramagnetism and diamagnetism in lecture 18. These are not required for doing the homework, but provide good intuition on how matter responds to external magnetic fields.

We then moved to Ohm's law and Faraday's law. The former is a phenomenological rule connecting electric fields and currents at the linear level. The latter, more nontrivially, governs the dynamics of electromagnetic fields, in contrast with the electro- and magnetostatics discussed in previous lectures.

Linear medium

Let's recapitulate the linear medium properties covered in the last discussion.

- Introduce the magnetization (intuitively the magnetic moment density)

$$\vec{M}(\vec{r}) \equiv \text{magnetic moment per volume}\,. \tag{1}$$

- Introducing a bound current density $\vec{J}_b(\vec{r'}) \equiv \vec{\nabla}'\times\vec{M}(\vec{r'})$ and a bound surface current $\vec{K}_b(\vec{r'}) \equiv \vec{M}(\vec{r'}) \times \hat{n}'$, we get the vector potential induced by the magnetization

$$\vec{A}_{\text{magnetization}}(\vec{r}) = \frac{\mu_0}{4\pi}\left(\int_{\mathcal{V}} d\tau' \frac{\vec{J}_b(\vec{r'})}{|\vec{r}-\vec{r'}|} + \oint_{\partial\mathcal{V}} da' \frac{\vec{K}_b(\vec{r'})}{|\vec{r}-\vec{r'}|}\right)\,. \tag{2}$$

- Now we introduce an auxiliary field as follows.
  - Starting with Ampère's law $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$, define the free current density $\vec{J}_f$ as $\vec{J}_f \equiv \vec{J} - \vec{J}_b$. This gives a current density decomposition

$$\vec{J} = \vec{J}_f + \vec{J}_b\,. \tag{3}$$

  - Plugging $\vec{J}_b = \vec{\nabla}\times\vec{M}$ into Ampère's law gives

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}_f + \mu_0\vec{\nabla}\times\vec{M} \quad\to\quad \vec{\nabla}\times\left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \vec{J}_f\,. \tag{4}$$

  - Introduce the auxiliary field

$$\vec{H} \equiv \frac{\vec{B}}{\mu_0} - \vec{M} \quad\to\quad \vec{\nabla}\times\vec{H} = \vec{J}_f\,. \tag{5}$$

  - Note $\vec{\nabla}\cdot\vec{B} = 0$ is still true.

- For linear media, we have a linear response relation

$$\vec{M} = \chi_m \vec{H} \tag{6}$$

where χ_m is called the magnetic susceptibility.

  – The magnetic field

$$\vec{B} = \mu_0 \left(\vec{H} + \vec{M}\right) = \mu_0 \left(\vec{H} + \chi_m \vec{H}\right) \quad \rightarrow \quad \vec{B} \equiv \mu \vec{H}, \tag{7}$$

  where $\mu \equiv \mu_0(1+\chi_m)$ is the permeability.

Ohm's law

The Ohm's law we learned in kindergarten reads $V = IR$, where V is the potential difference, I is the current, and R is the resistance. Accounting for the microscopic structure, namely the drag force due to scattering, one obtains the college version

$$\vec{J} = \sigma \vec{E}\,. \tag{8}$$

This states that the current density responds linearly to an external electric field, with the proportionality constant σ being the conductivity.

Faraday's law

A changing magnetic field induces an electric field.

- In differential form, this is Faraday's law

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}\,. \tag{9}$$

- In integral form

$$\oint_{\partial\Sigma} d\vec{l} \cdot \vec{E} = -\frac{d\Phi}{dt}, \quad \Phi \equiv \int_{\Sigma} d\vec{a} \cdot \vec{B}\,. \tag{10}$$

- Direction of $\vec{E}$ is such that if there were a wire on $\partial\Sigma$, the induced current would oppose the change in Φ (Lenz's law).

- Review the examples in lecture 19 to get a sense on how to compute the induced $\vec{E}$ in practice.

Homework 9

Homework 9 problems can be grouped as follows.

- Linear medium: problems 1-3.

- Ohm's law: problem 4.

- Faraday's law: problems 5-7.

Linear medium

Problem 1 A spherical shell of inner radius a and outer radius b is constructed of uniformly magnetized material with magnetization $\vec{M}$.

1a Find the equivalent volume and surface current distributions.

- Recall that for magnetic fields in matter, the bound volume current density is $\vec{J}_b = \vec{\nabla} \times \vec{M}$ and the bound surface current density is $\vec{K}_b = \vec{M} \times \vec{n}$.
  - In spherical coordinates, how to express $\hat{n}_{\text{outer}}$ and $\hat{n}_{\text{inner}}$ in terms of the basis $\{\hat{r}, \hat{\theta}, \hat{\phi}\}$?

1b Find the magnetic field $\vec{B}$ everywhere.

- Think of the spherical shell as a superposition of a uniformly magnetized ball of radius a with $\vec{M} = -M\hat{z}$ and another uniformly magnetized ball of radius b with $\vec{M} = M\hat{z}$.
- To find the magnetic field of each magnetized ball, recall example 5.11 and lecture 17 pp. 13-14.
  - Example 5.11: A spherical shell of radius R, carrying uniform surface charge σ, is set spinning at angular velocity $\vec{\omega}$. The magnetic field is

$$\vec{B}(\vec{r}) = \begin{cases} \frac{2}{3}\mu_0 \sigma R \vec{\omega} & r < R \\ \frac{2\mu_0 \omega R^4 \sigma}{3r^3} \left(\cos\theta \hat{r} + \frac{1}{2}\sin\theta\hat{\theta}\right) & r \geq R \end{cases} . \tag{11}$$

  - Using the argument in lecture 17, replacing $\sigma\omega$ with M/R gives the magnetic field of a uniformly magnetized ball

$$\vec{B}(\vec{r}) = \begin{cases} \frac{2}{3}\mu_0 \vec{M} & r < R \\ \frac{\mu_0 |\vec{m}_{\text{total}}|}{2\pi r^3} \left(\cos\theta \hat{r} + \frac{1}{2}\sin\theta\hat{\theta}\right) & r \geq R \end{cases} , \quad |\vec{m}_{\text{total}}| \equiv \frac{4\pi R^3}{3} M . \tag{12}$$

- Superpose the two magnetized balls with opposite magnetizations to get $\vec{B}$ everywhere.

1c Find the auxiliary field $\vec{H}$ everywhere.

- Use $\vec{H} = \vec{B}/\mu_0 - \vec{M}$ and the results from 1b.

Problem 2 Hint: Analyze the symmetry and use Ampère's law for the auxiliary field $\vec{\nabla} \times \vec{H} = \vec{J}_f$.

Problem 3 A solid sphere of radius R is made of a linear material with uniform magnetic susceptibility χ_m. It is placed in an otherwise uniform external field $\vec{B} = B_0\hat{z}$ (i.e. $\vec{B} = B_0\hat{z}$ at infinity). Note $\vec{\nabla} \times \vec{H} = \vec{J}_f = 0$ since there is no free current. Hence, we can write

$$\vec{H} = -\vec{\nabla} h \tag{13}$$

where h is a potential. Find $\vec{B}$ for $r > R$.

- This problem is telling you how to use h, the "scalar potential" for the auxiliary field $\vec{H}$, to solve boundary value problems in magnetostatics.
- Recall that in electrostatics, if $\rho = 0$, then Laplace's equation $\nabla^2 V = 0$ holds.
- Similarly, note $\vec{H} = -\vec{\nabla} h$ and $\vec{\nabla} \cdot \vec{H} = \vec{\nabla} \cdot \left(\vec{B}/\mu\right) = (\vec{\nabla} \cdot \vec{B})/\mu = 0$ where $\mu \equiv \mu_0(1 + \chi_m)$ is the magnetic permeability.
  - Plugging $\vec{H} = -\vec{\nabla} h$ into $\vec{\nabla} \cdot \vec{H} = 0$ leads to
  $$\nabla^2 h = 0 \,. \tag{14}$$
  Hence, the next thing to do is to solve this Laplace equation by writing down a general solution and identifying the boundary conditions.
  - This means for $r > R$, the general solution reads (why we keep $A_l r^l$ terms?)
  $$h(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta) \tag{15}$$
  and for $r < R$ that
  $$h(r, \theta) = \sum_{l=0}^{\infty} C_l r^l P_l(\cos\theta) \,. \tag{16}$$
- Boundary conditions.
  - At infinity
  $$\lim_{r\to\infty} \vec{H} = \frac{1}{\mu_0} \lim_{r\to\infty} \vec{B} = \frac{B_0}{\mu_0} \hat{z} \tag{17}$$
  which gives a boundary condition for h.
  - At $r = R$, we have
  $$\vec{\nabla} \times \vec{H} = 0 \quad \to \quad \vec{H}_\parallel(+) = \vec{H}_\parallel(-) \,, \tag{18}$$
  $$\vec{\nabla} \cdot \vec{B} = 0 \quad \to \quad \vec{B}_\perp(+) = \vec{H}_\perp(-) \,. \tag{19}$$
  - Translate these to boundary conditions for h.
  - These will suffice to solve h.
- The magnetic field $\vec{B}$ outside the sphere is then $\vec{B} = \mu_0 \vec{H} = -\mu_0 \vec{\nabla} h$.

Ohm's law

Problem 4 Two concentric metal spherical shells, of radius a and b, respectively, are separated by weakly conducting material of conductivity σ.

4a If they are maintained at a potential difference V, what current flows from one to the other?

- Assume the inner shell carries a fictitious surface charge σ_c.
- Find the electric field $\vec{E}$. Hint: Use symmetry to show $\vec{E} = E_r(r)\hat{r}$ then use Gauss's law.
- Solve $E_r(r)$ in terms of V. Ohm's law gives the current density $\vec{J} = \sigma \vec{E}$.
- Integrate over a sphere of radius r to get the current I as a function of r.
  - You will find I is a constant.

4b What is the resistance between the shells?

- Use the kindergarten version of Ohm's law $V = IR$ to find R.

Faraday's law

The strategies for solving the induced electric field $\vec{E}$ in these problems are similar. We usually start with computing the B-flux

$$\Phi \equiv \int_\Sigma d\vec{a} \cdot \vec{B}\,, \tag{20}$$

then use $\mathscr{E} = -d\Phi/dt$ to get the motional electromotive force (EMF). With symmetries we can find the induced electric field using

$$\mathscr{E} = \oint_{\partial\Sigma} d\vec{l} \cdot \vec{E}\,. \tag{21}$$

Problem 6 A metal bar of mass m slides frictionlessly on two parallel conducting rails a distance l apart. A resistor R is connected across the rails, and a uniform magnetic field $\vec{B}$, pointing into the page, fills the entire region.

6a If the bar moves to the right at speed v, what is the current in the resistor? In what direction does it flow?

- Find the motional EMF using

$$\mathscr{E} = -\frac{d\Phi}{dt}, \quad \Phi = \int_\Sigma d\vec{a} \cdot \vec{B}\,. \tag{22}$$

  As $\vec{B}$ is uniform, the B-flux is just $|\vec{B}|$ multiplied by the area enclosed by the circuit loop.

- The current is given by Ohm's law $\mathscr{E} = IR$. Its direction is determined by Lenz's law.

6b What is the magnetic force on the bar? In what direction?

- Consider an infinitesimal segment of the bar carrying charge dq with length dl.
- Use the Lorentz force formula $\vec{F} = \int dq \vec{v}_q \times \vec{B}$ to compute the magnetic force on the bar.

6d The initial kinetic energy of the bar was, of course, $\frac{1}{2}mv_0^2$. Check that the energy delivered to the resistor is exactly $\frac{1}{2}mv_0^2$.

- The kinetic energy is dissipated as resistor heats.
- Compute the total heat

$$Q = \int_0^\infty dt\, I^2 R \tag{23}$$

  and check that it equals $\frac{1}{2}mv_0^2$.

Problems 5 and 7 can be solved in a similar manner. Note the final answer in 5a contains $\arcsin$ or $\arccos$ functions.