

In this discussion we will first review inductance, Maxwell's equations in vacuum and in linear medium, and electromagnetic waves covered by lectures 19-22, then talk about homework 10. Handout PDFs are available at my personal website

<https://john137fs.github.io/jqzhang.github.io/teaching.html>

Review

Inductance

Consider two loops, say loop 1 and loop 2, carrying currents I_1 and I_2 , respectively. From the derivations in lecture 19 (pp. 14-16) we see that the magnetic flux Φ_1 through loop 1 receives two contributions: the flux induced by I_1 and the flux induced by I_2 .

- Using the Biot-Savart law we can show a linear relation

$$\Phi_1 = L_1 I_1 + M_{12} I_2, \quad (1)$$

with L_1 the self-inductance of loop 1 and M_{12} the mutual inductance between loop 1 and 2.

- Similarly, we can show that the magnetic flux Φ_2 through loop 2 is given by

$$\Phi_2 = L_2 I_2 + M_{21} I_1, \quad (2)$$

with L_2 the self-inductance of loop 2 and M_{21} the mutual inductance.

- The mutual inductance is symmetric: $M_{12} = M_{21} \equiv M$. It is determined by geometry:

$$M = M_{12} = M_{21} = \frac{\mu_0}{4\pi} \oint_{\text{loop 2}} \oint_{\text{loop 1}} \frac{d\vec{l}_1 \cdot d\vec{l}_2}{|\vec{r}_1 - \vec{r}_2|}. \quad (3)$$

- Energy stored in the magnetic field in an inductor is

$$W = \frac{1}{2} L I^2. \quad (4)$$

- To see this, note $dW = -dq \mathcal{E}$, then with Faraday's law $\mathcal{E} = -d\Phi/dt$ and the relation $\Phi = LI$ we have

$$\frac{dW}{dt} = -\frac{dq}{dt} \left(-\frac{d\Phi}{dt} \right) = I \frac{d\Phi}{dt} = LI \frac{dI}{dt} \rightarrow dW = d \left(\frac{1}{2} L I^2 \right). \quad (5)$$

- Setting the boundary condition $W|_{I=0} = 0$ gives $W = LI^2/2$.

Maxwell's equations

In this section we “derive” the Maxwell's equations. We want to generalize the equations derived for steady currents

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \quad \vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad (6)$$

to time-dependent source $\rho(t)$ and $\vec{J}(t)$ subject to the continuity equation

$$\frac{\partial \rho(t)}{\partial t} + \vec{\nabla} \cdot \vec{J}(t) = 0. \quad (7)$$

- Gauss's law is easily generalized to $\vec{\nabla} \cdot \vec{E} = \rho(t)/\epsilon_0$.
- The middle two equations in eq. (6) do not depend explicitly on source, therefore can be directly generalized.
- Ampère's law, however, requires a nontrivial generalization.
 - If we naively consider $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}(t)$, then taking divergence of both sides and using the continuity equation (7) yields

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 \vec{\nabla} \cdot \vec{J}(t) = -\mu_0 \frac{\partial \rho(t)}{\partial t}. \quad (8)$$

* Here LHS = $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \partial_i (\epsilon_{ijk} \partial_j B_k) = \epsilon_{ijk} \partial_i \partial_j B_k = 0$, but RHS is not necessarily zero, hence this naive generalization is wrong.

- We instead guess that Ampère's law generalizes to $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}(t) + \vec{Y}(t)$, then

$$0 = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 \vec{\nabla} \cdot \vec{J}(t) + \vec{\nabla} \cdot \vec{Y}(t) = -\mu_0 \frac{\partial \rho(t)}{\partial t} + \vec{\nabla} \cdot \vec{Y}(t). \quad (9)$$

* Use Gauss's law to get $\rho(t) = \epsilon_0 \vec{\nabla} \cdot \vec{E}$, then

$$\vec{\nabla} \cdot \vec{Y}(t) = \mu_0 \frac{\partial \rho(t)}{\partial t} = \vec{\nabla} \cdot \left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right), \quad (10)$$

which suggests $\vec{Y}(t) = \mu_0 \epsilon_0 (\partial \vec{E} / \partial t)$.

- Hence, we find a working generalization of the steady state equations that is compatible with the continuity equation

$$\boxed{\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \quad \vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}}. \quad (11)$$

This is the celebrated Maxwell equations.

- The integral form of the Maxwell equations can be useful. Integrating the fourth equation in (11) over a closed surface \mathcal{S} gives

$$\oint_{\mathcal{S}} d\vec{a} \cdot (\vec{\nabla} \times \vec{B}) = \oint_{\partial \mathcal{S}} d\vec{l} \cdot \vec{B} = \mu_0 \oint_{\mathcal{S}} d\vec{a} \cdot \vec{J} + \mu_0 \epsilon_0 \oint_{\mathcal{S}} d\vec{a} \cdot \frac{\partial \vec{E}}{\partial t} \quad (12)$$

where we have used the Stokes theorem.

Maxwell's equation in matter

In this section we consider how media effects modify the Maxwell equations. Keep in mind that to investigate electromagnetic phenomena in matter we usually need to consider the macroscopic average of the electromagnetic fields.

- This means the two equations:

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad (13)$$

remain unchanged after taking the macroscopic average.

- Recall that we used polarization \vec{P} (electric dipole density) and magnetization \vec{M} (magnetic dipole density) to describe the macroscopic response of media to external fields.

- This introduces bound charge and bound current

$$\rho_b = -\vec{\nabla} \cdot \vec{P}, \quad \vec{J}_b = \vec{\nabla} \times \vec{M}. \quad (14)$$

- For dynamic fields, we anticipate ρ_b and \vec{J}_b to be time-dependent.

- * A changing ρ_b leads to a corresponding current to guarantee bound charge conservation. This introduces a polarization current \vec{J}_P satisfying the continuity equation $\partial \rho_b / \partial t + \vec{\nabla} \cdot \vec{J}_P = 0$.

- * Plugging $\rho_b = -\vec{\nabla} \cdot \vec{P}$ into the above equation gives $\vec{J}_P = \partial \vec{P} / \partial t$.

- Hence, the free charge and free current are

$$\rho_f = \rho - \rho_b, \quad \vec{J}_f = \vec{J} - \vec{J}_b - \vec{J}_P. \quad (15)$$

- Use Gauss's law and Ampère's law to get $\rho = \epsilon_0 \vec{\nabla} \cdot \vec{E}$ and $\vec{J} = \vec{\nabla} \times \vec{B} / \mu_0$, then with equation (14) we find

$$\rho_f = \epsilon_0 \vec{\nabla} \cdot \vec{E} - \rho_b = \epsilon_0 \vec{\nabla} \cdot \vec{E} + \vec{\nabla} \cdot \vec{P} = \vec{\nabla} \cdot (\epsilon_0 \vec{E} + \vec{P}), \quad (16)$$

$$\vec{J}_f = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \vec{\nabla} \times \vec{M} - \frac{\partial \vec{P}}{\partial t} = \vec{\nabla} \times \left(\frac{\vec{B}}{\mu_0} - \vec{M} \right) - \frac{\partial \vec{P}}{\partial t}. \quad (17)$$

- Recall that we have defined the electric displacement \vec{D} and the auxiliary field \vec{H} as

$$\vec{D} \equiv \epsilon_0 \vec{E} + \vec{P}, \quad \vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}. \quad (18)$$

- This turns equations (16) and (17) into

$$\vec{\nabla} \cdot \vec{D} = \rho_f, \quad \vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{P}}{\partial t}. \quad (19)$$

- Equations (13) and (19) constitute the Maxwell equations in matter

$$\boxed{\vec{\nabla} \cdot \vec{D} = \rho_f, \quad \vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{P}}{\partial t}}. \quad (20)$$

- Boundary conditions remain unchanged

$$\begin{cases} D_{\perp}(+) - D_{\perp}(-) = \sigma_f \\ B_{\perp}(+) - B_{\perp}(-) = 0 \end{cases}, \quad \begin{cases} \vec{E}_{\parallel}(+) - \vec{E}_{\parallel}(-) = 0 \\ \vec{H}_{\parallel}(+) - \vec{H}_{\parallel}(-) = \vec{K}_f \times \hat{n} \end{cases}. \quad (21)$$

Energy flux

We review the electromagnetic energy flux result

$$\frac{dW}{dt} = -\frac{d}{dt} \int_{\mathcal{V}} d\tau U - \oint_{\partial\mathcal{V}} d\vec{a} \cdot \vec{S}$$

where W is the work done by electromagnetic force on the charge enclosed by \mathcal{V} , U is the energy density of the fields and \vec{S} is the Poynting vector given by

$$U = \frac{1}{2}\epsilon_0|\vec{E}|^2 + \frac{1}{2\mu_0}|\vec{B}|^2, \quad \vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}). \quad (22)$$

- The derivation can be found in lecture 21, pp. 8-9.
- If \mathcal{V} is chosen to be a region in which $\vec{J} = 0$, then $dW/dt = 0$. This leads to

$$\boxed{\frac{d}{dt} \int_{\mathcal{V}} d\tau U = - \oint_{\partial\mathcal{V}} d\vec{a} \cdot \vec{S}} \quad (23)$$

- This is intuitively the continuity equation for electromagnetic energy flux.

Electromagnetic waves in vacuum

In this section we utilize the Maxwell equations in vacuum to derive the electromagnetic wave equation and its solution.

- Let's first derive the wave equation.
 - In vacuum there is no source, so $\rho = 0$ and $\vec{J} = 0$.
 - The Maxwell equation (11) becomes

$$\vec{\nabla} \cdot \vec{E} = 0, \quad \vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \vec{\nabla} \times \vec{B} = \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t}. \quad (24)$$

- Note $\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E}$. On one hand

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t} \right) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) = -\mu_0\epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \quad (25)$$

and on the other hand

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\nabla^2 \vec{E}. \quad (26)$$

- * This gives

$$\frac{\partial^2 \vec{E}}{\partial t^2} - \frac{1}{\mu_0\epsilon_0} \nabla^2 \vec{E} = 0. \quad (27)$$

- * Similarly, we can derive

$$\frac{\partial^2 \vec{B}}{\partial t^2} - \frac{1}{\mu_0\epsilon_0} \nabla^2 \vec{B} = 0. \quad (28)$$

- Next, solve the wave equations (27) and (28).

- Consider the ansatz $\vec{E} = \vec{E}_0 f(x - vt)$ with v constant.
- Plugging this into the wave equation (27) returns $v = \pm 1/\sqrt{\mu_0 \epsilon_0}$. Here in SI units $c \equiv 1/\sqrt{\mu_0 \epsilon_0} = 3.00 \times 10^8$ m/s is the speed of light.
- We can verify that $\vec{B} = \vec{B}_0 f(x - vt)$ is a solution to equation (28).
- This means a \hat{x} -traveling solution to the Maxwell equations in vacuum is

$$\vec{E} = \vec{E}_0 f(x - ct), \quad \vec{B} = \vec{B}_0 f(x - ct), \quad (29)$$

where \vec{E}_0 and \vec{B}_0 subject to further restrictions given by the Maxwell equations.

- * Note $\vec{\nabla} \cdot \vec{E} = 0$ implies $0 = (\vec{E}_0 \cdot \hat{x}) \partial_x f$, giving $\vec{E}_0 \cdot \hat{x} = 0$. Similarly $\vec{\nabla} \cdot \vec{B} = 0$ gives $\vec{B}_0 \cdot \hat{x} = 0$.
- * Further, Faraday's law $\vec{\nabla} \times \vec{E} = -\partial \vec{B} / \partial t$ gives $\vec{B}_0 = \hat{x} \times \vec{E}_0 / c$.
- * Summarizing, electromagnetic waves in vacuum is transverse with $\vec{B}_0 \perp \vec{E}_0$. Specifically, for a \hat{x} -traveling wave

$$\vec{E}_0 \cdot \hat{x} = 0, \quad \vec{B}_0 \cdot \hat{x} = 0, \quad \vec{B}_0 = \frac{\hat{x} \times \vec{E}_0}{c}. \quad (30)$$

- For monochromatic waves traveling at \vec{k} , we can generalize the above results to write

$$\vec{E} = \vec{E}_0 \cos[\vec{k} \cdot \vec{x} - \omega t + \delta], \quad \vec{E}_0 \cdot \hat{k} = 0, \quad (31)$$

$$\vec{B} = \vec{B}_0 \cos[\vec{k} \cdot \vec{x} - \omega t + \delta], \quad \vec{B}_0 \cdot \hat{k} = 0, \quad (32)$$

$$\vec{B}_0 = \frac{\vec{k}}{\omega} \times \vec{E}_0, \quad \omega = c|\vec{k}|. \quad (33)$$

Homework 10

Homework 10 problems can be grouped as follows.

- Inductance: problems 1-2 and 4.
- Maxwell's equations: problems 3 and 5.
- Energy flux: problem 6.
- Electromagnetic waves: problem 7.

Inductance

Problem 1 Suppose in an infinitely long cable, surface current flows on an inner cylindrical conductor of radius a and on an outer cylindrical conductor of radius b . What is the self-inductance per unit length of this cable?

- First thing to consider is what do we mean by self-inductance of an infinitely long cable. What is the "loop" for defining self-inductance?

- The current flows from the inner cylinder to the outer cylinder, thus the loop is a rectangular plane of length l radially spanning the gap between the inner radius a and the outer radius b .
- The next task is to find the magnetic field between the cylinders to get the flux Φ_B passing through the loop.
- The self-inductance is then then given by $\Phi_B = LI$.
- Alternatively, we may attack this problem by considering the magnetic energy stored in the cylinders.
 - We may use

$$W = \frac{1}{2}LI^2 = \frac{1}{2\mu_0} \int d^3x |\vec{B}|^2 \quad (34)$$

to find the self-inductance.

Problem 2 Two coils have self-inductances L_1 and L_2 and mutual inductance M . They carry currents I_1 and I_2 , respectively.

2a Show that the stored magnetic field energy is given by

$$W = \frac{1}{2}L_1I_1^2 + \frac{1}{2}L_2I_2^2 + MI_1I_2. \quad (35)$$

- The idea is to use $dW = -dq_1 \mathcal{E}_1 - dq_2 \mathcal{E}_2$ with the motional EMFs given by

$$\mathcal{E}_1 = -\frac{d\Phi_1}{dt}, \quad \mathcal{E}_2 = -\frac{d\Phi_2}{dt} \quad (36)$$

and

$$\Phi_1 = L_1I_1 + MI_2, \quad \Phi_2 = L_2I_2 + MI_1. \quad (37)$$

2b Suppose the mutual inductance of the system varies with distance x : i.e. $M = M(x)$. Find the force on the coils in terms of $dM(x)/dx$. Note that as in the case of calculating the force between capacitor plates from energy considerations, you must be careful as to how you calculate the work in a virtual displacement. i.e. The answer is NOT simply

$$F_x = -\frac{\partial W}{\partial x} = -\frac{dM}{dx}I_1I_2 \quad (\text{this might be wrong})$$

Two cases are particularly simple: 1) constant current can be established by applying a suitable EMF as the coils are moved - external work from EMF agent is involved 2) constant flux can be established by turning off any external EMF agent, allowing the the current to vary such as to keep the flux constant. Compute using method 2 (constant flux method). This latter mode is called the "persistent mode", and it is often used in operating a superconducting coil.

- To find the force, we start with

$$F_x = -\frac{\partial W}{\partial x} = -\frac{\partial W}{\partial M} \frac{dM}{dx}. \quad (38)$$

- Now we need to find $\partial W/\partial M$ subject to the constant flux constraint (can be achieved in e.g. superconductors)

$$\Phi_1 = L_1 I_1 + M I_2 = \text{constant}, \quad \Phi_2 = L_2 I_2 + M I_1 = \text{constant}. \quad (39)$$

- Note this constraint means $\{I_1, I_2\}$ are expressible in terms of $\{\Phi_1, \Phi_2, L_1, L_2, M\}$, which implies I_1 and I_2 depend on M , then $\partial I_1/\partial M$ and $\partial I_2/\partial M$ are expected to be nonzero.
- So the partial derivative

$$\frac{\partial W}{\partial M} = \frac{\partial}{\partial M} \left(\frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M I_1 I_2 \right) \quad (40)$$

contains $\partial I_1/\partial M$ and $\partial I_2/\partial M$ terms.

- To summarize, to find F_x , we need to calculate $\partial W/\partial M$, which needs $\partial I_1/\partial M$ and $\partial I_2/\partial M$.
 - To find $\partial I_1/\partial M$ and $\partial I_2/\partial M$, we need to solve equation (39) to express $\{I_1, I_2\}$ in terms of $\{\Phi_1, \Phi_2, L_1, L_2, M\}$.
- If you are familiar with linear algebra, we may use a more elegant method to solve this problem. See appendix A (not required).

Problem 4 Consider a RLC circuit. The capacitor does not have any charge at the beginning and that $(R/L)^2 - 4/(LC) > 0$.

4a Suppose the switch is closed at $t = 0$. Compute $I(t)$ through the resistor.

- Note

$$\mathcal{E}_0 - L \frac{dI}{dt} - \frac{Q}{C} - IR = 0. \quad (41)$$

- Taking derivative, we get

$$-L \frac{d^2 I}{dt^2} - \frac{I}{C} - \frac{dI}{dt} R = 0. \quad (42)$$

- Initial condition is $I|_{t=0} = 0$ such that $L(dI/dt)$ is finite. Solve the above differential equation with the initial condition should give you $I(t)$.
 - Hint: consider the trial solution $I(t) = I_0 e^{st}$.

Maxwell's equations

Problem 3 A fat wire, radius a , carries a constant current I , uniformly distributed over its cross section. A narrow gap in the wire, of width $w \ll a$, forms a parallel-plate capacitor. Find the magnetic field in the gap, at a distance $s < a$ from the axis.

- This can be solved using the integral form of the Maxwell equation

$$\oint_S d\vec{a} \cdot (\vec{\nabla} \times \vec{B}) = \oint_{\partial S} d\vec{l} \cdot \vec{B} = \mu_0 \oint_S d\vec{a} \cdot \vec{J} + \mu_0 \epsilon_0 \oint_S d\vec{a} \cdot \frac{\partial \vec{E}}{\partial t}. \quad (43)$$

- You may put the surface S in the gap where $\vec{J} = 0$. The above simplifies to

$$\oint_{\partial S} d\vec{l} \cdot \vec{B} = \mu_0 \epsilon_0 \oint_S d\vec{a} \cdot \frac{\partial \vec{E}}{\partial t}. \quad (44)$$

- In cylindrical coordinates, from symmetry we know \vec{B} only has $\hat{\phi}$ component. We also know $\vec{E} = (\sigma/\epsilon_0)\hat{z}$ and $I = dQ/dt$ with $Q = \sigma\pi a^2$.
- Plugging these into equation (44) should give you the magnetic field in the gap.
- Comment: this is how a varying electric field induces a magnetic field.

Energy flux

Problem 6 A very long solenoid of radius a , with n turns per unit length, carries a current I_s . Coaxial with the solenoid, at radius $b \gg a$, is a circular ring of wire, with resistance R . When the current in the solenoid is (gradually) decreased, a current I_r is induced in the ring.

6a Calculate I_r , in terms of dI_s/dt .

- The physics here is a changing I_s induces a changing flux passing through the ring, thereby generates a motional EMF \mathcal{E} that gives rise to I_r .
- Ohm's law gives $I_r = \mathcal{E}/R$ and Faraday's law gives

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}(B_{\text{solenoid}}\pi a^2). \quad (45)$$

- This should give you I_r in terms of dI_s/dt and other parameters. Note here $dI_s/dt < 0$, be careful with the sign.

6b The power ($I_r^2 R$) delivered to the ring must have come from the solenoid. Confirm this by calculating the Poynting vector just outside the solenoid (the electric field is due to the changing flux in the solenoid; the magnetic field is due to the current in the ring). Integrate over the entire surface of the solenoid, and check that you recover the correct total power.

- We want to compute the power given by the electromagnetic energy flux

$$P = \int_{\text{solenoid}} d\vec{a} \cdot \vec{S} = \int_{\text{solenoid}} d\vec{a} \cdot \frac{1}{\mu_0} (\vec{E} \times \vec{B}). \quad (46)$$

- The magnetic field just outside the solenoid is generated by the ring, given by

$$\vec{B} \approx \frac{\mu_0 I_r}{2} \frac{b^2}{(b^2 + z^2)^{3/2}} \hat{z} \quad (47)$$

where we have used the approximation $b \gg a$.

- The electric field just outside the solenoid is generated by the changing flux in the solenoid. The integral form of Faraday's law gives ¹

$$\oint d\vec{l} \cdot \vec{E} = -\frac{d\Phi}{dt} = I_r R. \quad (48)$$

This helps you find \vec{E} .

- Now computing the integral (46) should give you $I_r^2 R$.

Electromagnetic waves

Problem 7b Write down the (real) electric and magnetic fields for a monochromatic plane wave of amplitude E_0 , frequency ω , and phase angle zero that is traveling in the direction from the origin to the point $(1, 1, 1)$, with polarization parallel to the xz plane. In each case, sketch the wave, and give the explicit Cartesian components of \vec{k} and \hat{n} .

- We know the polarization direction \hat{n} of \vec{E}_0 is parallel to the xz plane. This means $\hat{n} \cdot \hat{y} = 0$. So suppose

$$\hat{n} = n_x \hat{x} + n_z \hat{z}. \quad (49)$$

- The propagation direction is

$$\hat{k} = \frac{1}{\sqrt{3}}(1, 1, 1). \quad (50)$$

- As electromagnetic waves are transverse, we must have $\hat{k} \cdot \hat{n} = 0$, giving

$$0 = \hat{k} \cdot \hat{n} = \frac{1}{\sqrt{3}}(n_x + n_z) = 0. \quad (51)$$

- Choosing $n_z > 0$ (you have the freedom to choose the sign of n_z), we can determine \hat{n} . This means $\vec{E}_0 = E_0 \hat{n}$.
- The magnetic field amplitude is then $\vec{B} = \vec{k} \times \vec{E}_0 / \omega = \hat{k} \times \vec{E}_0 / c$.
- These lead to

$$\vec{E} = \vec{E}_0 \cos \left[\vec{k} \cdot \vec{x} - \omega t \right], \quad (52)$$

$$\vec{B} = \vec{B}_0 \cos \left[\vec{k} \cdot \vec{x} - \omega t \right]. \quad (53)$$

¹Here we have neglected the flux Φ_r induced by the magnetic field from the ring. This is because as $\Phi_{\text{solenoid}} \propto a^2$ and $\Phi_{\text{ring}} \propto I_r a^2 / b \propto a^4 / b$, $\Phi_{\text{ring}} / \Phi_{\text{solenoid}}$ is suppressed by the geometry $b \gg a$.

A Problem 2b, linear algebra approach

- First, notice W can be written as a quadratic form

$$W = \frac{1}{2} I^T O I, \quad I \equiv \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}, \quad O \equiv \begin{pmatrix} L_1 & M \\ M & L_2 \end{pmatrix}. \quad (54)$$

Note here O is symmetric, i.e., $O^T = O$.

- The constant flux constraint (39) translates to

$$\Phi = O I, \quad \Phi \equiv \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}. \quad (55)$$

– This means the current vector is solved by $I = O^{-1} \Phi$, then $I^T = \Phi^T (O^{-1})^T = \Phi^T O^{-1}$.

- We rewrite W in terms of Φ to utilize the $\Phi = \text{constant}$ constraint

$$W = \frac{1}{2} (\Phi^T O^{-1}) O (O^{-1} \Phi) = \frac{1}{2} \Phi^T O^{-1} \Phi. \quad (56)$$

– As $\Phi = \text{constant}$, we have

$$\frac{\partial W}{\partial M} = \frac{1}{2} \Phi^T \frac{\partial O^{-1}}{\partial M} \Phi = \frac{1}{2} I^T O^T \frac{\partial O^{-1}}{\partial M} O I = \frac{1}{2} I^T O \frac{\partial O^{-1}}{\partial M} O I. \quad (57)$$

– To calculate $O \frac{\partial O^{-1}}{\partial M} O$, we notice the identity $O^{-1} O = \mathbb{I}_{2 \times 2}$, then taking derivative on both sides yields

$$\frac{\partial O^{-1}}{\partial M} O + O^{-1} \frac{\partial O}{\partial M} = 0 \quad \rightarrow \quad \frac{\partial O^{-1}}{\partial M} O = -O^{-1} \frac{\partial O}{\partial M}. \quad (58)$$

– This gives

$$O \frac{\partial O^{-1}}{\partial M} O = -O O^{-1} \frac{\partial O}{\partial M} = -\frac{\partial O}{\partial M}. \quad (59)$$

- Hence,

$$\frac{\partial W}{\partial M} = \frac{1}{2} I^T O \frac{\partial O^{-1}}{\partial M} O I = \boxed{-\frac{1}{2} I^T \frac{\partial O}{\partial M} I}. \quad (60)$$

Simplifying this should suffice to give you F_x .