PHY322 S26
Discussion 13
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This discussion is devoted to reviewing lectures 22-25 and discussing homework 11. You are welcome to come to my office (Chamberlin 5214) to ask questions any weekday before the final. The door is open as long as I'm in my office.

Review

Plane wave

Here we review the complex notation for electromagnetic waves in vacuum and their energy density and flux. Next, we discuss how to generalize these results to electromagnetic waves in linear medium.

- We have derived the sinusoidal representations for monochromatic electromagnetic waves in *vacuum* propagating at direction $\vec{k}$ with polarization $\hat{n}$:

$$\vec{E} = \vec{E}_0 \cos\left[\vec{k}\cdot\vec{x} - \omega t + \delta\right], \quad \vec{E}_0 \cdot \hat{k} = 0\,, \tag{1}$$

$$\vec{B} = \vec{B}_0 \cos\left[\vec{k}\cdot\vec{x} - \omega t + \delta\right], \quad \vec{B}_0 \cdot \hat{k} = 0\,, \tag{2}$$

$$\vec{B}_0 = \frac{\vec{k}}{\omega} \times \vec{E}_0, \quad \omega = c|\vec{k}|\,. \tag{3}$$

  - Mathematically, this is equivalent to the following complex notation

$$\vec{E} = \mathrm{Re}[\tilde{\vec{E}}], \quad \vec{B} = \mathrm{Re}[\tilde{\vec{B}}]\,, \tag{4}$$

$$\boxed{\tilde{\vec{E}} = \tilde{E}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}\hat{n}, \quad \tilde{\vec{B}} = \frac{\vec{k}}{\omega}\times\vec{E}; \quad \tilde{E}_0 = |\tilde{E}_0|e^{i\delta}, \quad \hat{n}\cdot\hat{k} = 0, \quad \omega = c|\vec{k}|\,.} \tag{5}$$

- The energy density $U(t,\vec{x})$ for monochromatic wave in vacuum (propagating at $\hat{k}$) is

$$U(t,\vec{x}) = \frac{1}{2}\left(\epsilon_0|\vec{E}|^2 + \frac{1}{\mu_0}|\vec{B}|^2\right) = \epsilon_0|\vec{E}_0|^2\cos^2\left(\vec{k}\cdot\vec{x} - \omega t + \delta\right)\,, \tag{6}$$

  and the energy flux, given by the Poynting vector $\vec{S}(t,\vec{x})$, is

$$\vec{S}(t,\vec{x}) = \frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right) = c\,U(t,\vec{x})\hat{k}\,. \tag{7}$$

  - In applications, the time scale τ of concern is usually much larger than the period of the wave $T = 2\pi/\omega$, i.e., $\tau \gg T$. This means we are interested in the time averages

$$\langle U\rangle \equiv \frac{1}{T}\int_0^T dt\, U(t,\vec{x}) = \frac{1}{2}\epsilon_0|\vec{E}_0|^2\,, \quad \text{intensity} \equiv \langle|\vec{S}|\rangle = c\,\langle U\rangle\ . \tag{8}$$

- Now we generalize the above results to linear media with permittivity ϵ and permeability μ.

- The sourceless Maxwell equations in matter reads ($\rho_f = 0$ and $\vec{J}_f = 0$)

$$\vec{\nabla} \cdot \vec{D} = 0, \quad \vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} . \tag{9}$$

- Use the linear relations $\vec{D} = \epsilon \vec{E}$ and $\vec{H} = \vec{B}/\mu$ to get

$$\vec{\nabla} \cdot \vec{E} = 0, \quad \vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \vec{\nabla} \times \vec{B} = \mu\epsilon \frac{\partial \vec{E}}{\partial t} . \tag{10}$$

- Staring at the above equation, we note its solution can be obtained by replacing $\mu_0\epsilon_0$ with $\mu\epsilon$ in the vacuum solutions.
- This gives monochromatic electromagnetic waves in linear media propagating at direction $\vec{k}$ with polarization $\hat{n}$:

$$\vec{E} = \vec{E}_0 \cos\left[\vec{k}\cdot\vec{x} - \omega t + \delta\right], \quad \vec{E}_0 \cdot \hat{k} = 0\,, \tag{11}$$

$$\vec{B} = \vec{B}_0 \cos\left[\vec{k}\cdot\vec{x} - \omega t + \delta\right], \quad \vec{B}_0 \cdot \hat{k} = 0\,, \tag{12}$$

$$\vec{B}_0 = \frac{\vec{k}}{\omega} \times \vec{E}_0, \quad \omega = v|\vec{k}|, \quad v \equiv \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{n}, \quad n \equiv \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} \tag{13}$$

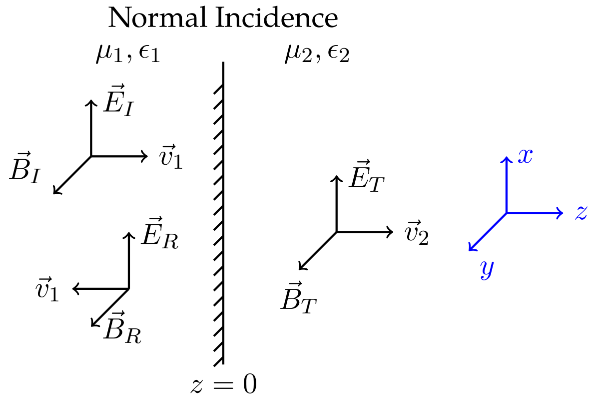
  with n the index of refraction.
- Energy density and intensity for monochromatic waves in linear media are given by

$$\langle U \rangle = \frac{1}{2}\epsilon|\vec{E}_0|^2, \quad \langle|\vec{S}|\rangle = \frac{1}{2}\epsilon v|\vec{E}_0|^2 = v\,\langle U \rangle\ . \tag{14}$$

Reflection and transmission

In this section, we consider how EM waves are reflected and transmitted when propagating through the interface of two linear media. We will start with normal incidence, then generalize the results to oblique incidence.

Normal incidence Consider two linear media with ϵ_1, μ_1 and ϵ_2, μ_2 filling the $z < 0$ and $z > 0$ regions, respectively. A plane wave of frequency ω traveling along the $\hat{z}$ direction with polarization at $\hat{x}$ approaches the interface from left to right, as shown below.



- In complex notation, the incident wave, reflected wave and transmitted wave are given by

$$\begin{cases}\vec{\tilde{E}}_I = \tilde{E}_{0I} e^{i(k_1 z - \omega t)}\hat{x} \\ \vec{\tilde{B}}_I = \frac{\tilde{E}_{0I}}{v_1} e^{i(k_1 z - \omega t)}\hat{y}\end{cases} \quad \begin{cases}\vec{\tilde{E}}_R = \tilde{E}_{0R} e^{i(-k_1 z - \omega t)}\hat{x} \\ \vec{\tilde{B}}_R = \frac{-\tilde{E}_{0R}}{v_1} e^{i(-k_1 z - \omega t)}\hat{y}\end{cases} \quad \begin{cases}\vec{\tilde{E}}_T = \tilde{E}_{0T} e^{i(k_2 z - \omega t)}\hat{x} \\ \vec{\tilde{B}}_T = \frac{\tilde{E}_{0T}}{v_2} e^{i(k_2 z - \omega t)}\hat{y}\end{cases} \tag{15}$$

  - Note here the reflected and transmitted waves are guesses.
  - Reflected and transmitted waves have the same polarization and the same ω.

- The relations among the reflected and transmitted waves and the incident wave are fixed by the boundary conditions

$$D_\perp(+) = D_\perp(-), \quad B_\perp(+) = B_\perp(-), \quad \vec{E}_\parallel(+) = \vec{E}_\parallel(-), \quad \vec{H}_\parallel(+) = \vec{H}_\parallel(-)\,. \tag{16}$$

  - At $z = 0$, this gives

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}\,, \quad \frac{\tilde{E}_{0I}}{\mu_1 v_1} - \frac{\tilde{E}_{0R}}{\mu_1 v_1} = \frac{\tilde{E}_{0T}}{\mu_2 v_2}\,. \tag{17}$$

  - Solving the above equations, we find

$$\tilde{E}_{0R} = \frac{1-\beta}{1+\beta}\tilde{E}_{0I}\,, \quad \tilde{E}_{0T} = \frac{2}{1+\beta}\tilde{E}_{0I}\,, \quad \beta \equiv \frac{\mu_1}{\mu_2}\frac{n_2}{n_1}\,. \tag{18}$$

  - For $\mu_1 \approx \mu_2 \approx \mu_0$, the above relation becomes

$$\tilde{E}_{0R} = \frac{n_1 - n_2}{n_1 + n_2}\tilde{E}_{0I}\,, \quad \tilde{E}_{0T} = \frac{2n_1}{n_1 + n_2}\tilde{E}_{0I}\,. \tag{19}$$

  - The reflection and transmission coefficients are defined w.r.t. intensities (14)

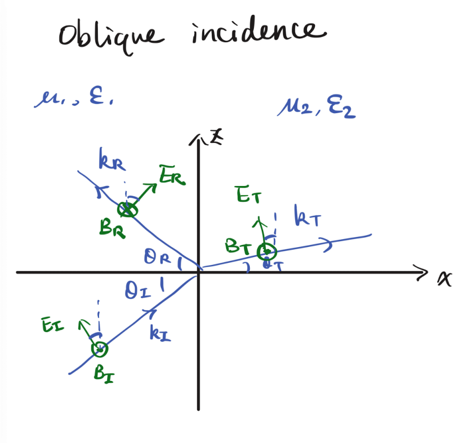
$$R \equiv \frac{I_R}{I_I}\,, \quad T \equiv \frac{I_T}{I_I}\,; \quad I = \langle|\vec{S}|\rangle \propto \epsilon v |\vec{E}|^2\,. \tag{20}$$

    - This leads to

$$R = \frac{\epsilon_1 v_1}{\epsilon_1 v_1}\frac{|\tilde{E}_R|^2}{|\tilde{E}_I|^2} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2, \quad T = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\frac{|\tilde{E}_T|^2}{|\tilde{E}_I|^2} = \frac{4n_1 n_2}{(n_1 + n_2)^2}\,. \tag{21}$$

    - One can verify that $R + T = 1$, confirming energy conservation.

Oblique incidence We have derived the reflection and transmission coefficients for normal incidence waves. Now we want to generalize the previous results to an oblique incidence as shown below.



- Parametrize the incident, reflected and transmitted waves

$$\begin{cases} \tilde{\vec{E}}_I = \tilde{\vec{E}}_{0I} e^{i(\vec{k}_I \cdot \vec{r} - \omega t)} \\ \tilde{\vec{B}}_I = \frac{1}{v_1} (\hat{k}_I \times \tilde{\vec{E}}_I) \end{cases} \quad \begin{cases} \tilde{\vec{E}}_R = \tilde{\vec{E}}_{0R} e^{i(\vec{k}_R \cdot \vec{r} - \omega t)} \\ \tilde{\vec{B}}_R = \frac{1}{v_1} (\hat{k}_R \times \tilde{\vec{E}}_R) \end{cases} \quad \begin{cases} \tilde{\vec{E}}_T = \tilde{\vec{E}}_{0T} e^{i(\vec{k}_T \cdot \vec{r} - \omega t)} \\ \tilde{\vec{B}}_T = \frac{1}{v_2} (\hat{k}_T \times \tilde{\vec{E}}_T) \end{cases} \tag{22}$$

$$k_I v_1 = k_R v_1 = k_T v_2 = \omega \, . \tag{23}$$

- Exercise the boundary conditions at $z = 0$:

$$D_\perp(+) = D_\perp(-), \quad B_\perp(+) = B_\perp(-), \quad \vec{E}_\parallel(+) = \vec{E}_\parallel(-), \quad \vec{H}_\parallel(+) = \vec{H}_\parallel(-) \, . \tag{24}$$

  - Matching the exponentials, we find

    $$\vec{k}_I \cdot \vec{r}\Big|_{z=0} = \vec{k}_R \cdot \vec{r}\Big|_{z=0} = \vec{k}_T \cdot \vec{r}\Big|_{z=0} \, , \tag{25}$$

    giving

    $$(k_I)_x = (k_R)_x = (k_T)_x, \quad (k_I)_y = (k_R)_y = (k_T)_y \, . \tag{26}$$

  - In our convention $(k_I)_y = 0$, which means $(k_R)_y = (k_T)_y = 0$. Hence $\vec{k}_I$, $\vec{k}_R$ and $\vec{k}_T$ form a plane (in our convention, the xz plane).
  - Note $(k_I)_x = (k_R)_x = (k_T)_x$ implies

    $$k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T \, . \tag{27}$$

    From equation (23) we see that
    - ∗ In medium 1, $k_I = k_R = \omega/v_1$, which yields $\theta_I = \theta_R$.
    - ∗ In medium 2, $k_T = \omega/v_2$, which gives $k_I/k_T = (\omega/v_1)/(\omega/v_2) = v_2/v_1 = n_1/n_2$, therefore

      $$\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2} \, . \tag{28}$$

  - We summarize the above results as three geometric optical rules
    1. The wave vectors $\vec{k}_I$, $\vec{k}_R$ and $\vec{k}_T$ form a plane.
    2. The angle of incidence equals the angle of reflection, $\theta_I = \theta_R$.
    3. Snell's law $n_1 \sin\theta_I = n_2 \sin\theta_T$.

- Our next task is to find the reflection and transmission coefficients for oblique incidence.
  - For a detailed derivation, see lecture 23 pp. 10-14.
  - We used the boundary conditions

    $$\begin{cases} D_\perp(+) = D_\perp(-) \\ B_\perp(+) = B_\perp(-) \\ \vec{E}_\parallel(+) = \vec{E}_\parallel(-) \\ \vec{H}_\parallel(+) = \vec{H}_\parallel(-) \end{cases} \rightarrow \begin{cases} \epsilon_1(\tilde{\vec{E}}_{0I} + \tilde{\vec{E}}_{0R})_z = \epsilon_2(\tilde{\vec{E}}_{0T})_z \\ (\tilde{\vec{B}}_{0I} + \tilde{\vec{B}}_{0R})_z = (\tilde{\vec{B}}_{0T})_z \\ (\tilde{\vec{E}}_{0I} + \tilde{\vec{E}}_{0R})_{x,y} = (\tilde{\vec{E}}_{0T})_{x,y} \\ \frac{1}{\mu_1}(\tilde{\vec{B}}_{0I} + \tilde{\vec{B}}_{0R})_{x,y} = \frac{1}{\mu_2}(\tilde{\vec{B}}_{0T})_{x,y} \end{cases} \quad \tilde{\vec{B}}_0 = \frac{\hat{k} \times \tilde{\vec{E}}_0}{v} \, . \tag{29}$$

  - Suppose the polarization $\hat{n}$ is on the xy plane. We obtained

    $$\boxed{R = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2, \quad T = \frac{4\alpha\beta}{(\alpha + \beta)^2} \, ; \quad \alpha \equiv \frac{\cos\theta_T}{\cos\theta_I}, \quad \beta \equiv \frac{\mu_1 n_2}{\mu_2 n_1} \, .} \tag{30}$$

  - In homework 11 problem 2 you are asked to compute the reflection and transmission coefficients for a different polarization.

Wave guide

In this section we consider how electromagnetic waves propagate inside a metal pipe.

- To obtain the wave solution inside the pipe traveling along the $\hat{z}$ direction, we focus on the vacuum region and solve the Maxwell equations with certain boundary conditions.
- The vacuum Maxwell equations read

$$\vec{\nabla}\cdot\vec{E}=0,\quad \vec{\nabla}\cdot\vec{B}=0,\quad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t},\quad \vec{\nabla}\times\vec{B}=\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}\,. \tag{31}$$

- The boundary conditions on the inner boundary of the pipe are

$$\vec{E}_{\parallel}(+)-\vec{E}_{\parallel}(-)=0\quad\rightarrow\quad \vec{E}_{\parallel}\Big|_{\text{inner wall}}=0\,, \tag{32}$$

$$B_{\perp}(+)-B_{\perp}(-)=0\quad\rightarrow\quad B_{\perp}|_{\text{inner wall}}=0\,. \tag{33}$$

- Consider wave solutions traveling along the $\hat{z}$ direction of the form

$$\tilde{\vec{E}}=\tilde{\vec{E}}_0(x,y)e^{i(kz-\omega t)},\quad \tilde{\vec{B}}=\tilde{\vec{B}}_0(x,y)e^{i(kz-\omega t)}\,. \tag{34}$$

  - Plugging them into the Maxwell equations gives

$$\frac{\partial\tilde{E}_{0z}}{\partial y}-ik\tilde{E}_{0y}=i\omega\tilde{B}_{0x},\quad ik\tilde{E}_{0x}-\frac{\partial\tilde{E}_{0z}}{\partial x}=i\omega\tilde{B}_{0y}\,, \tag{35}$$

$$\frac{\partial\tilde{B}_{0z}}{\partial y}-ik\tilde{B}_{0y}=-\frac{i\omega}{c^2}\tilde{E}_{0x},\quad ik\tilde{B}_{0x}-\frac{\partial\tilde{B}_{0z}}{\partial x}=-\frac{i\omega}{c^2}\tilde{E}_{0y}\,. \tag{36}$$

  - After some manipulations (lecture 25, pp. 1-3) we obtain two differential equations

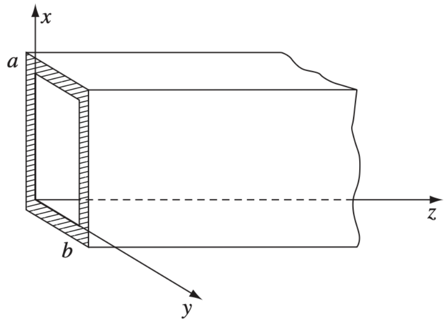
$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\frac{\omega}{c}\right)^2-k^2\right]\tilde{E}_{0z}(x,y)=0\,, \tag{37}$$

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\frac{\omega}{c}\right)^2-k^2\right]\tilde{B}_{0z}(x,y)=0\,. \tag{38}$$

    - We define TE and TM waves as

$$\begin{cases}\tilde{E}_{0z}=0 & \text{TE (transverse electric) waves}\\ \tilde{B}_{0z}=0 & \text{TM (transverse magnetic) waves}\end{cases}\,. \tag{39}$$

- We considered TE wave modes in a rectangular wave guide with the geometry shown below (credit: Griffiths' electrodynamics)



- We want to solve the TE wave modes with $\tilde{E}_{0z} = 0$ and

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right] \tilde{B}_{0z}(x,y) = 0 \tag{40}$$

  with the boundary condition $B_\perp|_{\text{inner wall}} = 0$.
- Use separation of variables, $\tilde{B}_{0z}(x,y) = X(x)Y(y)$ and $B_\perp|_{\text{inner wall}} = 0$ to get

$$\tilde{B}_{0z}(x,y) = B_{mn} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \tag{41}$$

$$\tilde{B}_{0x} = \frac{ik}{(\omega/c)^2 - k^2} \frac{\partial \tilde{B}_{0z}}{\partial x}, \quad \tilde{B}_{0y} = \frac{ik}{(\omega/c)^2 - k^2} \frac{\partial \tilde{B}_{0z}}{\partial y} \tag{42}$$

$$\tilde{\vec{B}} = \tilde{\vec{B}}_0(x,y) e^{i(kz-\omega t)}, \quad k = \sqrt{\left(\frac{\omega}{c}\right)^2 - \pi^2 \left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]}. \tag{43}$$

- Let $\omega_{mn} \equiv c\pi\sqrt{(m/a)^2 + (n/b)^2}$. Note if $\omega < \omega_{mn}$ then k becomes imaginary, for which the TE modes decay exponentially due to e^{-kz}.
- Note the phase velocity $v = \omega/k$ can be greater than c, but the group velocity $v_g = d\omega/dk$ is always smaller than the speed of light.
- The electric fields for the TE modes are given by

$$\tilde{E}_{0x} = \frac{i\omega}{\left(\frac{\omega}{c}\right)^2 - k^2} \frac{\partial \tilde{B}_{0z}}{\partial y}, \quad \tilde{E}_{0y} = \frac{-i\omega}{\left(\frac{\omega}{c}\right)^2 - k^2} \frac{\partial \tilde{B}_{0z}}{\partial x}. \tag{44}$$

Homework 11

Homework 11 problems can be grouped as follows.

- Maxwell's equations: problems 1 and 6.
- Oblique incidence: problem 2.
- Complex notation for electromagnetic waves: problem 3.
- Wave guide: problems 4-5.

Maxwell's equations

Problem 1 Comment: This problem is about showing the gauge symmetry in the vacuum Maxwell equations explicitly. The math is straightforward and the physics is interesting.

Problem 6 Comment: This problem is about the retarded potential solutions to the sourced Maxwell equations in the Lorenz gauge. Similar to problem 1, the math is straightforward. Just be patient.

Oblique incidence

Problem 2 We analyzed in lecture, the oblique incidence scattering with the polarization in the plane of incidence. In this problem, analyze the case of polarization perpendicular to the plane of incidence (i.e. electric fields in the $\hat{y}$ direction in the figure). You need to

1. Obtain $\tilde{E}_{0R}$ and $\tilde{E}_{0T}$ as a function of $\{\tilde{E}_{0T}, \alpha, \beta\}$ where α and β were defined in class. As usual, you can use any results given in lecture.
2. Sketch $\tilde{E}_{0R}/\tilde{E}_{0I}$ and $\tilde{E}_{0T}/\tilde{E}_{0I}$ as functions of θ_I for the case $\beta = n_2/n_1 = 1.5$ (parameters for a common glass) assuming $\mu_1 = \mu_2$. (Note that for this β is always π out of phase.)
3. Confirm that your Fresnel equations reduce to the proper forms at normal incidence.
4. Compute the reflection and transmission coefficients, and check that they sum to unity.

We may mimic the derivation for the oblique incidence in the review section to solve this problem.

- We start with parametrizing the incident, reflected and transmitted waves

$$\begin{cases} \vec{\tilde{E}}_I = \vec{\tilde{E}}_{0I} e^{i(\vec{k}_I\cdot\vec{r}-\omega t)} \\ \vec{\tilde{B}}_I = \frac{1}{v_1}(\hat{k}_I \times \vec{\tilde{E}}_I) \end{cases} \quad \begin{cases} \vec{\tilde{E}}_R = \vec{\tilde{E}}_{0R} e^{i(\vec{k}_R\cdot\vec{r}-\omega t)} \\ \vec{\tilde{B}}_R = \frac{1}{v_1}(\hat{k}_R \times \vec{\tilde{E}}_R) \end{cases} \quad \begin{cases} \vec{\tilde{E}}_T = \vec{\tilde{E}}_{0T} e^{i(\vec{k}_T\cdot\vec{r}-\omega t)} \\ \vec{\tilde{B}}_T = \frac{1}{v_2}(\hat{k}_T \times \vec{\tilde{E}}_T) \end{cases} \tag{45}$$

$$k_I v_1 = k_R v_1 = k_T v_2 = \omega\,. \tag{46}$$

- Use the boundary conditions (29)

$$\begin{cases} \epsilon_1(\vec{\tilde{E}}_{0I} + \vec{\tilde{E}}_{0R})_z = \epsilon_2(\vec{\tilde{E}}_{0T})_z \\ (\vec{\tilde{B}}_{0I} + \vec{\tilde{B}}_{0R})_z = (\vec{\tilde{B}}_{0T})_z \\ (\vec{\tilde{E}}_{0I} + \vec{\tilde{E}}_{0R})_{x,y} = (\vec{\tilde{E}}_{0T})_{x,y} \\ \frac{1}{\mu_1}(\vec{\tilde{B}}_{0I} + \vec{\tilde{B}}_{0R})_{x,y} = \frac{1}{\mu_2}(\vec{\tilde{B}}_{0T})_{x,y} \end{cases} \qquad \vec{\tilde{B}}_0 = \frac{\hat{k} \times \vec{\tilde{E}}_0}{v} \tag{47}$$

  with the specific geometry to solve the problem.

Complex notation for electromagnetic waves

Problem 3 Quadratic in complex representation of a wave. Suppose $f(\vec{r}, t) = A\cos(\vec{k}\cdot\vec{r} - \omega t + \delta_a)$ and $g(\vec{r}, t) = B\cos(\vec{k}\cdot\vec{r} - \omega t + \delta_b)$. Show that $\langle fg\rangle = (1/2)\mathrm{Re}(\tilde{f}\tilde{g}^*)$, where the star denotes complex conjugation and the $\langle\ldots\rangle$ is time average. [Note that this only works if the two waves have the same $\vec{k}$ and ω, but they need not have the same amplitude or phase.] For example,

$$\langle u\rangle = \frac{1}{4}\mathrm{Re}\left(\varepsilon_0 \vec{\tilde{E}}\cdot\vec{\tilde{E}}^* + \frac{1}{\mu_0}\vec{\tilde{B}}\cdot\vec{\tilde{B}}^*\right) \quad \text{and} \quad \langle\vec{S}\rangle = \frac{1}{2\mu_0}\mathrm{Re}\left(\vec{\tilde{E}}\times\vec{\tilde{B}}^*\right).$$

- To verify the identity $\langle fg\rangle = (1/2)\mathrm{Re}(\tilde{f}\tilde{g}^*)$ we compute LHS and RHS respectively.
- For LHS, we note

$$\langle fg\rangle = AB\left\langle \cos(\vec{k}\cdot\vec{r} - \omega t + \delta_a)\cos(\vec{k}\cdot\vec{r} - \omega t + \delta_b)\right\rangle\,. \tag{48}$$

  - Use the hint

$$\cos\left(\vec{k}\cdot\vec{r}-\omega t+\delta_a\right)\cos\left(\vec{k}\cdot\vec{r}-\omega t+\delta_b\right)=\frac{1}{2}\left[\cos\left(2\vec{k}\cdot\vec{r}-2\omega t+\delta_a+\delta_b\right)+\cos\left(\delta_a-\delta_b\right)\right] \tag{49}$$

    to compute the time average.

- For RHS, we use

$$\tilde{f}=Ae^{i(\vec{k}\cdot\vec{r}-\omega t+\delta_a)},\quad \tilde{g}=Be^{i(\vec{k}\cdot\vec{r}-\omega t+\delta_b)} \tag{50}$$

  to calculate $(1/2)\mathrm{Re}(\tilde{f}\tilde{g}^*)$.

- You should find LHS = RHS.

Wave guide

Problem 4 Consider the energy flow in a rectangular wave guide for a TE_{mn} mode.

4a Compute the power flowing down the waveguide

$$\Delta P\equiv\int_{\text{cross section}}d\vec{a}\cdot\langle\vec{S}\rangle\,. \tag{51}$$

- The time-average of the Poynting vector is

$$\langle\vec{S}\rangle=\frac{1}{\mu_0}\left\langle\vec{E}\times\vec{B}\right\rangle=\frac{1}{2\mu_0}\mathrm{Re}\left(\tilde{\vec{E}}\times\tilde{\vec{B}}^*\right) \tag{52}$$

  where we have used the result from problem 3.

  - For a TE_{mn} mode, we have

$$\tilde{B}_{0x}=\frac{ik}{(\omega/c)^2-k^2}\frac{\partial\tilde{B}_{0z}}{\partial x},\quad \tilde{B}_{0y}=\frac{ik}{(\omega/c)^2-k^2}\frac{\partial\tilde{B}_{0z}}{\partial y} \tag{53}$$

$$\tilde{E}_{0x}=\frac{i\omega}{\left(\frac{\omega}{c}\right)^2-k^2}\frac{\partial\tilde{B}_{0z}}{\partial y},\quad \tilde{E}_{0y}=\frac{-i\omega}{\left(\frac{\omega}{c}\right)^2-k^2}\frac{\partial\tilde{B}_{0z}}{\partial x}\,. \tag{54}$$

  - The cross section is $d\vec{a}=dxdy\hat{z}$.
  - You should get

$$\Delta P=\frac{1}{2\mu_0}\frac{\omega k}{\left[\left(\frac{\omega}{c}\right)^2-k^2\right]}\int dxdy\left[\frac{\partial\tilde{B}_{0z}}{\partial y}\frac{\partial\tilde{B}^*_{0z}}{\partial y}+\frac{\partial\tilde{B}_{0z}}{\partial x}\frac{\partial\tilde{B}^*_{0z}}{\partial x}\right]\,. \tag{55}$$

  - We also know that for a TE_{mn} mode

$$\tilde{B}_{0z}(x,y)=B_{mn}\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) \tag{56}$$

$$k=\frac{1}{c}\sqrt{\omega^2-\omega_{mn}^2},\quad \omega_{mn}\equiv c\pi\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}\,. \tag{57}$$

    Plugging these into equation (55) gives you the final answer.

4b Problem 4b is very similar to problem 4b. Hint: use

$$u\rangle=\frac{1}{4}\mathrm{Re}\left(\varepsilon_0\tilde{\vec{E}}\cdot\tilde{\vec{E}}^*+\frac{1}{\mu_0}\tilde{\vec{B}}\cdot\tilde{\vec{B}}^*\right)\,. \tag{58}$$