PHY322 S26
Discussion 2
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In this discussion we focus on solving the homework problems with **symmetry**.

Homework 2

The homework problems can be grouped as follows:

- Prob. 1&2: Evaluation exercises where the spherical symmetry greatly simplifies the work.
- Prob. 3&5: Delta function calculations.
- Prob. 4: Vector function decomposition.
- Prob. 6: Electric field induced by a spherically symmetric charge distribution.

Problems 1 and 2

1a We evaluate $\nabla^2 f$ where

$$f(x,y,z) = \ln\left(x^2+y^2+z^2+a^2\right)\ . \tag{1}$$

- Staring at the f function, we note the argument $x^2+y^2+z^2$ is invariant under 3d rotations.
  - In spherical coordinates (r,θ,ϕ), the rotational symmetry implies that $x^2+y^2+z^2=r^2$ is independent of θ and ϕ.
  - This means that θ and ϕ degrees of freedom in f are reduced by symmetry.
  - Hence, $f(x,y,z)\to f(r,\theta,\phi)=\ln(r^2+a^2)$.
- Work in spherical coordinates. The Laplacian reads

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}\,. \tag{2}$$

  - What are $\partial f/\partial\theta$ and $\partial f/\partial\phi$? Find a symmetry argument.
  - What is $\nabla^2 f$ then?

2a & 2b In these evaluation exercises we would encounter computing the derivatives of the unit vectors $\{\hat r,\hat\theta,\hat\phi\}$ in spherical coordinates. In lec. 2 (pp. 11-13) we have learnt that

$$\hat r = \sin\theta\cos\phi\hat x+\sin\theta\sin\phi\hat y+\cos\theta\hat z \tag{3}$$
$$\hat\theta = \cos\theta\cos\phi\hat x+\cos\theta\sin\phi\hat y-\sin\theta\hat z \tag{4}$$
$$\hat\phi = -\sin\phi\hat x+\cos\phi\hat y\,, \tag{5}$$

from which you can derive

$$\frac{\partial\hat r}{\partial\theta}=\hat\theta,\quad \frac{\partial\hat r}{\partial\phi}=\sin\theta\hat\phi,\quad \frac{\partial\hat\theta}{\partial\theta}=-\hat r,\quad \frac{\partial\hat\theta}{\partial\phi}=\cos\theta\hat\phi,\quad \frac{\partial\hat\phi}{\partial\theta}=0,\quad \frac{\partial\hat\phi}{\partial\phi}=-\left(\sin\theta\hat r+\cos\theta\hat\theta\right)\,. \tag{6}$$

With these expressions, 2a is then straightforward, but 2b is tedious. Patience is the key.

2c We compute explicitly

$$\vec{\nabla} \times \left[\frac{\vec{x} - \vec{x'}}{|\vec{x} - \vec{x'}|^3} \right] \tag{7}$$

with $\vec{\nabla}$ acting on $\vec{x}$.

- First, translate the origin to $\vec{x'}$ and define $\vec{R} \equiv \vec{x} - \vec{x'}$. It follows that $\vec{\nabla}_R = \vec{\nabla}$ (why?) and
$$\vec{\nabla} \times \left[\frac{\vec{x} - \vec{x'}}{|\vec{x} - \vec{x'}|^3} \right] = \vec{\nabla}_R \times \left(\frac{\vec{R}}{|\vec{R}|^3} \right) . \tag{8}$$
- Consider a similar but different problem. How to describe the symmetry of $\vec{r} = r\hat{r}$?
  - Note that in spherical coordinates $\vec{r}$ only has radial component.
  - What is the *intuition* of the curl of $\vec{r}$ then (lec. 2)?
    - $*$ Applying the Stokes theorem to $\vec{r}$, we get
    $$\int_S d\vec{a} \cdot \left(\vec{\nabla} \times \vec{r} \right) = \oint_{\partial S} d\vec{l} \cdot \vec{r} . \tag{9}$$
    - $*$ Let S be a circle centered about the origin. The loop integral $\oint_{\partial S} d\vec{l} \cdot \vec{r}$ vanishes because $\vec{r}$ only has radial component. Hence, the integral
    $$\int_S d\vec{a} \cdot \left(\vec{\nabla} \times \vec{r} \right) = 0 , \tag{10}$$
    meaning that $\vec{r}$ has no circulation.
    - $*$ It then looks very likely that $\vec{\nabla} \times \vec{r}$ vanishes.
  - Now how to check this intuition rigorously?
    - $*$ Use the $\vec{\nabla} \times \vec{r}$ expression in spherical coordinates.
- We then compute $\vec{\nabla}_R \times \left(\vec{R}/|\vec{R}|^3 \right)$ explicitly by using the curl expression of a vector field.
- The result is zero because the vector field $\vec{R}/|\vec{R}|^3$ has no circulation for the reason above.

Problems 3 and 5

5 We compute the integral

$$I \equiv \int dxdydz \delta^{(3)} \left(\vec{C}(x) \right) \vec{F}(\vec{x}) \tag{11}$$

over all space where

$$\vec{C}(x) = (x-1)y\hat{x} + (y-3)\hat{y} + xy(z^2 - 4)\hat{z} \tag{12}$$

$$\vec{F} = x^2\hat{x} + y\hat{y} + z\hat{z} . \tag{13}$$

- The key idea is to use the theorem (lec. 3)
$$\int_D d\tau h(\vec{x}) \delta^{(3)} \left(\vec{f}(x) \right) = \sum_n \frac{h\left(\vec{x}_{*n}\right)}{\left| \det \left(\frac{\partial f_i}{\partial x_j} \right) \right|_{\vec{x} = \vec{x}_{*n}}} \tag{14}$$
where $\vec{x}_{*n}$ are the zeros of $\vec{f}$ contained in D.

- To apply the theorem, we need the zeros of $\vec{C}(\vec{x})$ and the Jacobian(s) $\det\left(\partial C_i/\partial x_j\right)_{\vec{x}=\vec{x}_{*n}}$.
  - There are two zeros of $\vec{C}(\vec{x})$ as it is quadratic in z.
  - The explicit expression for the Jacobian is

$$\det\left(\frac{\partial C_i}{\partial x_j}\right)_{\vec{x}=\vec{x}_{*n}} = \det\begin{pmatrix} \frac{\partial C_x}{\partial x} & \frac{\partial C_x}{\partial y} & \frac{\partial C_x}{\partial z} \\ \frac{\partial C_y}{\partial x} & \frac{\partial C_y}{\partial y} & \frac{\partial C_y}{\partial z} \\ \frac{\partial C_z}{\partial x} & \frac{\partial C_z}{\partial y} & \frac{\partial C_z}{\partial z} \end{pmatrix}_{\vec{x}=\vec{x}_{*n}} . \tag{15}$$

Problem 4

This problem is to verify that for a vector function $\vec{F}$

$$\vec{\nabla}\times\vec{F}=0 \quad \to \quad \vec{F}=\vec{\nabla}S \tag{16}$$

$$\vec{\nabla}\cdot\vec{F}=0 \quad \to \quad \vec{F}=\vec{\nabla}\times\vec{A}\,. \tag{17}$$

The nontrivial parts are 4c and 4d. We will discuss 4d as an example.

- Straightforward computations show that $\vec{\nabla}\cdot\vec{F}_1=0$ and $\vec{\nabla}\times\vec{F}_1\neq 0$.
- Hence, $\vec{F}_1=\vec{\nabla}\times\vec{A}=x^2\hat{z}$ for a vector field $\vec{A}$. We want to solve $\vec{A}$ that satisfies

$$\partial_x A_y - \partial_y A_x = x^2 \tag{18}$$

$$\partial_y A_z - \partial_z A_y = 0 \tag{19}$$

$$\partial_z A_x - \partial_x A_z = 0\,. \tag{20}$$

- In lec. 3, we have seen that one can rewrite $\vec{A}$ as $\vec{A}=\vec{G}+\vec{\nabla}H$ without changing $\vec{F}_1$.
  - This means that $\vec{F}_1=\vec{\nabla}\times\vec{A}$ is invariant under the transformation $\{\vec{A}\to\vec{A}+\vec{\nabla}H\}$, which is usually referred to as gauge symmetry.
  - With this symmetry, we can manually set one component of $\vec{A}$ to be at any value achievable via a gauge transformation.
  - For example, we can set $A_x=0$, then eqs. (18-20) reduce to

$$\partial_x A_y = x^2, \quad \partial_y A_z - \partial_z A_y = 0, \quad \partial_x A_z = 0\,. \tag{21}$$

  - Integrating the above equations, we will be able to get all components of $\vec{A}$ such that $\vec{F}_1=\vec{\nabla}\times\vec{A}$.

Problem 6

We compute the electric field of a spherical surface of radius R with an isotropic charge density σ using the expression

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x'\,\frac{\rho(\vec{x'})(\vec{x}-\vec{x'})}{|\vec{x}-\vec{x'}|^3}\,. \tag{22}$$

- First thing to notice is the **rotational symmetry** of the charge distribution.

  - The charge distribution $\rho(\vec{x'}) = \sigma\delta(|\vec{x'}| - R)$ is invariant under rotations about $\hat{z}$.
  - Note that $\vec{E}$ inherits this rotational symmetry as can be seen from the Coulomb law (22).
  - At $(0, 0, z)$, the E_x and E_y components vanish, as they violate the rotational symmetry.
  - Hence, we only need to compute the radial component E_z at $(0, 0, z)$.

- Next, we carry out the integral at $\vec{x} = (0, 0, z)$ step by step.
  - From left to right, we will need d^3x', $\rho(\vec{x'})$ and $\vec{x} - \vec{x'}$.
  - Because of the rotational symmetry, we use spherical coordinates $\vec{x'} = (r', \theta', \phi') = r'\hat{r'}$.
  - We now have $d^3x' = r'^2 dr' \sin\theta' d\theta' d\phi'$ and $\rho(\vec{x'}) = \sigma\delta(r' - R)$.
  - The z-component of the integrand

$$\left[\frac{\rho(\vec{x'})(\vec{x} - \vec{x'})}{|\vec{x} - \vec{x'}|^3}\right]_z = \frac{\sigma\delta(r' - R)(z - r'\cos\theta')}{(z^2 + r'^2 - 2zr'\cos\theta')^{3/2}} \,. \tag{23}$$

  - The z-component of the electric field

$$E_z = \frac{1}{4\pi\epsilon_0}\int_0^\infty r'^2 dr' \int_0^{2\pi} \sin\theta' d\theta' \int_0^{2\pi} d\phi' \frac{\sigma\delta(r' - R)(z - r'\cos\theta')}{(z^2 + r'^2 - 2zr'\cos\theta')^{3/2}} \,. \tag{24}$$

  - The r' and ϕ' integrals are straightforward.
  - For the θ' integral, we may use

$$\int_{\theta'=0}^{\theta'=2\pi} \sin\theta' d\theta' f(\cos\theta') = \int_{\cos\theta'=-1}^{\cos\theta'=1} d\cos\theta' f(\cos\theta') \,. \tag{25}$$

- After obtaining the result, we should check the answer from other perspectives.
  - In lec. 4, you have calculated

$$V = \frac{\sigma R}{\epsilon_0}\begin{cases} 1 & r < R \\ \frac{R}{r} & r > R \end{cases} . \tag{26}$$

    Is this compatible with your $\vec{E}$ in the sense that $\vec{E} = -\vec{\nabla}V$?
  - Is your answer compatible with the Gauss theorem $\oint d\vec{a} \cdot \vec{E} = \int d^3x'(\rho(\vec{x'})/\epsilon_0)$?

Supplement materials

More on Levi-Civita symbol

We show how to utilize the Levi-Civita symbol to prove the vector identity appeared in HW1 problem 1

$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \vec{B}\left(\vec{A} \cdot \vec{C}\right) - \vec{C}\left(\vec{A} \cdot \vec{B}\right) . \tag{27}$$

Through this example you may see the convenience and elegance of the Einstein summation conventions and the Levi-Civita symbol introduced in the first discussion.

Recall that the Levi-Civita symbol in $\mathbb{R}^3$ is defined as

$$\epsilon_{ijk} \equiv \begin{cases} 1 & (i,j,k) \in \{(1,2,3),(3,1,2),(2,3,1)\} \\ -1 & (i,j,k) \in \{(1,3,2),(3,2,1),(2,1,3)\} \\ 0 & \text{otherwise} \end{cases} \tag{28}$$

with which the i-th component of the vector cross product $\vec{A} \times \vec{B}$ can be written as

$$\left(\vec{A} \times \vec{B}\right)_i = \epsilon_{ijk} A_j B_k . \tag{29}$$

We expand the i-th component of the left hand side of eq. (27)

$$\left[\vec{A} \times \left(\vec{B} \times \vec{C}\right)\right]_i = \epsilon_{ijk} A_j \left(\vec{B} \times \vec{C}\right)_k = \epsilon_{ijk} A_j \epsilon_{klm} B_l C_m = \epsilon_{ijk}\epsilon_{klm} A_j B_l C_m . \tag{30}$$

Note that the anti-symmetry in ϵ_{klm} implies

$$\epsilon_{klm} = -\epsilon_{lkm} = \epsilon_{lmk} , \tag{31}$$

giving (rf. discussion 1 eq. (24))

$$\epsilon_{ijk}\epsilon_{klm} = \epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} . \tag{32}$$

Hence, eq. (30) reduces to

$$\begin{aligned} \left[\vec{A} \times \left(\vec{B} \times \vec{C}\right)\right]_i &= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) A_j B_l C_m & (33) \\ &= \delta_{il}\delta_{jm} A_j B_l C_m - \delta_{im}\delta_{jl} A_j B_l C_m & (34) \\ &= B_i A_m C_m - C_i A_l B_l . & (35) \end{aligned}$$

Noting $A_m C_m = \vec{A} \cdot \vec{C}$ and $A_l B_l = \vec{A} \cdot \vec{B}$, we obtain

$$\left[\vec{A} \times \left(\vec{B} \times \vec{C}\right)\right]_i = B_i \left(\vec{A} \cdot \vec{C}\right) - C_i \left(\vec{A} \cdot \vec{B}\right) \tag{36}$$

which means

$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \vec{B}\left(\vec{A} \cdot \vec{C}\right) - \vec{C}\left(\vec{A} \cdot \vec{B}\right) . \tag{37}$$

The merit of this approach is that by using the Einstein summation convention and the Levi-Civita symbol we are able to manipulate many terms simultaneously.

As another example, we show that (rf. Griffith front-page (11))

$$\nabla \times \left(\nabla \times \vec{A}\right) = \nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2 \vec{A} \tag{38}$$

We start with rewriting the left hand side using the Levi-Civita symbol

$$\left[\nabla \times \left(\nabla \times \vec{A}\right)\right]_i = \epsilon_{ijk}\partial_j \left(\nabla \times \vec{A}\right)_k = \epsilon_{ijk}\partial_j \left(\epsilon_{klm}\partial_l A_m\right) = \epsilon_{ijk}\epsilon_{klm}\partial_j\partial_l A_m \tag{39}$$

where we have moved ϵ_{klm} out of the ∂_j derivative since the components of the Levi-Civita symbol are constants. Using the identity (32) again, we obtain

$$\begin{aligned} \left[\nabla \times \left(\nabla \times \vec{A}\right)\right]_i &= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\partial_j\partial_l A_m && (40) \\ &= \delta_{il}\delta_{jm}\partial_j\partial_l A_m - \delta_{im}\delta_{jl}\partial_j\partial_l A_m && (41) \\ &= \partial_m\partial_i A_m - \partial_j\partial_j A_i && (42) \end{aligned}$$

What are the complete forms of the two terms in the last line? Note

$$\partial_m\partial_i A_m \to \sum_{m=1}^{3} \partial_m\partial_i A_m = \partial_i\partial_m A_m = \partial_i\left(\nabla \cdot \vec{A}\right) \tag{43}$$

$$\partial_j\partial_j A_i \to \sum_{m=1}^{3} \partial_j\partial_j A_i = \nabla^2 A_i \tag{44}$$

We find

$$\left[\nabla \times \left(\nabla \times \vec{A}\right)\right]_i = \partial_i\left(\nabla \cdot \vec{A}\right) - \nabla^2 A_i \tag{45}$$

which proves eq. (38).