PHY322 S26
Discussion 3
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In this discussion, we will start with reviewing lectures 3-6, then talk about homework 3.

Review

In this section, we review the relations between source (charge distribution) and field (electric field) in the context of electrostatics. A sketch of lectures 3-6 is as follows.

- In electrostatics, all charges are at rest → charge density $\rho(\vec{x})$ is time-independent (static).
- Experiments tell us that the electric field $\vec{E}(\vec{x})$ generated by a static source $\rho(\vec{x'})$ is
  $$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x'\,\frac{\rho(\vec{x'})(\hat{x}-\hat{x'})}{|\vec{x}-\vec{x'}|^2}\,, \tag{1}$$
  which is called Coulomb's law.
  - Coulomb's law implies two differential equations
    $$\boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}, \quad \vec{\nabla}\times\vec{E} = 0}\,. \tag{2}$$
  - The curl-less condition implies the existence of a scalar potential V for $\vec{E}$ such that
    $$\vec{E} = -\vec{\nabla}V\,. \tag{3}$$
  - Plugging this into $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$ yields Poisson's equation
    $$\boxed{-\nabla^2 V = \frac{\rho}{\epsilon_0}}\,. \tag{4}$$
- Conversely, if we start with the differential equations (2) rather than Coulomb's law, we can reconstruct Coulomb's law as long as appropriate boundary conditions are imposed.
  - Eq. (2) with the boundary condition
    $$\lim_{|\vec{x}|\to\infty}\vec{E}(\vec{x}) = 0, \quad \lim_{|\vec{x}|\to\infty}|\vec{x}|^2\rho(\vec{x}) = 0 \tag{5}$$
    has a unique solution (up to ambiguities in V_0) that is equivalent to Coulomb's law
    $$\boxed{\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x'\,\frac{\rho(\vec{x'})(\hat{x}-\hat{x'})}{|\vec{x}-\vec{x'}|^2}, \quad V(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x'\,\frac{\rho(\vec{x'})}{|\vec{x}-\vec{x'}|} + V_0}\,. \tag{6}$$
- Boundary conditions (problem 5b and 5d)
  - Normal component
    $$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \quad\Rightarrow\quad E^{\perp}(+) - E^{\perp}(-) = \frac{\sigma}{\epsilon_0}\,. \tag{7}$$
  - Parallel component
    $$\vec{\nabla}\times\vec{E} = 0 \quad\Rightarrow\quad E^{\parallel}(+) - E^{\parallel}(-) = 0\,. \tag{8}$$
- Ideal conductor is an object of equipotential ($V =$ constant) that can carry charge.

Homework 3

The homework problems can be classified as follows.

- Given charge distribution, find electric field/potential (prob. 1,3,4,5,6,7).
- Given electric field/potential, find charge distribution (prob. 2).
- Boundary condition exercises (prob. 5).

Throughout all the problems, symmetry can simplify the work, especially when using Gauss's law.

From charge to field (prob. 1,3,4,5,6,7)

In these problems, you are given charge distributions with certain symmetries, and your task is to figure out what is the induced electric field. Two typical strategies are

- Identify symmetry → choose coordinates → find $\vec{E}$ using Gauss's law or Coulomb's law.
- Integrate Poisson's equation $-\nabla^2 V = \rho/\epsilon_0$ to get V → find $\vec{E}$ using $\vec{E} = -\vec{\nabla} V$.

We will see that in many situations, symmetry arguments can tell us that certain components of the electric field are vanishing. By identifying the symmetry structures you will also be able to obtain intuitions about electrostatic systems. For the remaining non-vanishing components that cannot be completely fixed by symmetry, we usually use Coulomb's law to compute them.

Problem 1 (Poisson's equation) The charge distribution considered in this problem is a sphere of radius R with uniform charge density ρ.

- Poisson's equation in spherical coordinates reads

$$-\nabla^2 V = \begin{cases} \frac{\rho}{\epsilon_0} & r \le R \\ 0 & r > R \end{cases} . \tag{9}$$

- Spherical symmetry implies that the electric potential $V = V(r)$ is independent of θ and ϕ.
  - The Poisson equation then simplifies to

$$-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = \begin{cases} \frac{\rho}{\epsilon_0} & r \le R \\ 0 & r > R \end{cases} . \tag{10}$$

  - Integrating the above equation with respect to r, you should be able to get $\partial V/\partial r$.
  - In spherical coordinates, the electric field

$$\vec{E} = -\vec{\nabla} V = -\frac{\partial V}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi} . \tag{11}$$

    - What are E_θ and E_ϕ? Why?
    - What is E_r?

Problem 3 (Gauss's law) In this problem the charge distribution is given by

$$\rho(z) = \rho_0 e^{-\frac{|z|}{t}}, \quad \rho_0, t = \text{constant}. \tag{12}$$

- What are the symmetries in the charge distribution?
  - The charge density $\rho(z)$ is independent of x and y, and satisfies $\rho(z) = \rho(-z)$.
- What are the symmetries in the electric field $\vec{E}$?
  - Given the symmetries in the charge density, what are E_x and E_y?
  - What does the symmetry $\rho(z) = \rho(-z)$ tell you about the property of E_z? (Hint: the transformation $z \to -z$ could be viewed as a π-rotation about the x axis.)
- With these symmetries, we can use the integral form of Gauss's law

  $$\oint_{\partial V} d\vec{a} \cdot \vec{E} = \int_V d^3x \frac{\rho}{\epsilon_0} \tag{13}$$

  to figure out $\vec{E}$.
- The potential V is then obtained by integrating $\vec{E} = -\vec{\nabla} V$.
- In the region $|z| \gg t$, $\rho(z) \approx 0$ due to the exponential suppression $e^{-|z|/t}$.
  - Effectively, $\vec{E}$ in the region $|z| \gg t$ should behave as if there were a uniform sheet of charge with effective surface charge density σ_{eff}.
  - On one hand, you know $\vec{E}$ from Gauss's law. On the other hand, you know that $\vec{E}$ in $|z| \gg t$ effectively behaves as $|\vec{E}| = \sigma_{\text{eff}}/(2\epsilon_0)$. Comparing them gives you σ_{eff}.

Problem 4 (Coulomb's law) Here you are asked to find the electric potential and field induced by a uniformly charged cylinder of length L, radius R, and charge density ρ.

- The charge distribution has cylindrical symmetry (rotational symmetry about the z axis).
  - Work in cylindrical coordinates (s, ϕ, z).
- The charge distribution is known

  $$\rho(s, \phi, z) = \begin{cases} \rho & s \le R \text{ and } -L/2 \le z \le L/2 \\ 0 & \text{otherwise} \end{cases}. \tag{14}$$

  - Therefore, we use Coulomb's law to find the electric potential at $\vec{x} = (0, 0, z > L/2)$

    $$V(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{\rho(\vec{x'})}{|\vec{x} - \vec{x'}|} = \frac{1}{4\pi\epsilon_0} \int_{-L/2}^{L/2} dz' \int_0^R ds' s' \int_0^{2\pi} d\phi' \frac{\rho}{\sqrt{(z - z')^2 + s'^2}}. \tag{15}$$

  - The ϕ' integration is obvious. For the z' and s' integrations, you may use $ds's' = \left(ds'^2\right)/2$ to integrate over s'^2 first, then integrate over z'.
- The electric field

  $$\vec{E} = -\vec{\nabla} V = -\frac{\partial V}{\partial s}\hat{s} - \frac{1}{s}\frac{\partial V}{\partial \phi}\hat{\phi} - \frac{\partial V}{\partial z}\hat{z}. \tag{16}$$

  - According to cylindrical symmetry, what are $\partial V/\partial s$ and $\partial V/\partial \phi$ at $(0, 0, z > L/2)$?
  - How to compute $\partial V/\partial z$?

Problem 6 (Coulomb's law) Here we will discuss how to compute $\vec{E}_{\text{other}}(+)$ as a function of ε.

- The charge distribution is known, so we compute $\vec{E}_{\text{other}}(+)$ using Coulomb's law

$$\vec{E}_{\text{other}}(+) = \frac{1}{4\pi\epsilon_0} \int da' \frac{(\hat{x} - \hat{x'})\sigma}{|\vec{x} - \vec{x'}|^2}, \tag{17}$$

- In spherical coordinates centered about the origin, $\vec{x} = R\hat{z}$, $\vec{x'} = R\hat{r}$, and

$$da' = R^2 d\theta \sin\theta d\phi, \quad \vec{x} - \vec{x'} = R\hat{z} - R\hat{r}, \quad |\vec{x} - \vec{x'}| = \sqrt{2R^2 - 2R^2\cos\theta}\,. \tag{18}$$

- We shall compute (note the lower limit of the θ integral is ε)

$$\vec{E}_{\text{other}}(+) = \frac{\sigma}{4\pi\epsilon_0} R^2 \int_0^{2\pi} d\phi \int_\varepsilon^\pi d\theta \sin\theta \frac{R(\hat{z} - \hat{r})\sqrt{2R^2 - 2R^2\cos\theta}}{2R^2 - 2R^2\cos\theta}\,. \tag{19}$$

  - Rotational symmetry → $\vec{E}_{\text{other}}(+)$ only has z-component → only need $\vec{E}_{\text{other}}(+) \cdot \hat{z}$.
  - Use the usual change of variable $d\theta \sin\theta \to -d\cos\theta$.

- What is $\vec{E}_{\text{other}}(+)$ in the limit $a/R \to 0$?

Problem 7 The integration structure is very similar to that in problem 6.

- Hint: from lecture 6, we see that the repel force here is given by the expression

$$\vec{F} = \int_{\text{hemisphere}} da \vec{f}, \quad \vec{f} = \frac{\sigma^2}{2\epsilon_0}\hat{n}\,. \tag{20}$$

- In spherical coordinates, we know that
  - The area element $da = R^2 d\theta \sin\theta d\phi$, and the normal vector $\hat{n} = \hat{r}$.
  - Using rotational symmetry, what is the direction of $\vec{F}$?

From field to charge

Problem 2 (Poisson's equation) In this problem you are asked to compute the charge distribution that gives arise to the potential (time-averaged potential of a neutral hydrogen atom)

$$V(r) = \frac{q_e}{4\pi\epsilon_0} \frac{e^{-2\alpha r}}{r}(1 + \alpha r) \tag{21}$$

where q_e is a constant charge (the e notation in the HW) and $\alpha = 1/a_0$ with a_0 the Bohr radius.

- The potential here is very similar to that of a point charge, so we rewrite

$$V(r) = \frac{u(r)}{r}, \quad u(r) \equiv \frac{q_e}{4\pi\epsilon_0} e^{-2\alpha r}(1 + \alpha r)\,. \tag{22}$$

- Poisson's equation tells us that

$$\rho = -\epsilon_0 \nabla^2 V = -\epsilon_0 \nabla^2 \left(\frac{u(r)}{r}\right). \tag{23}$$

  - In computing this object you will encounter $u(r)\nabla^2(1/r)$.
  - How to express $\nabla^2(1/r)$ in terms of a delta function? (this corresponds to the discrete charge distribution)