

In this discussion, we will start with reviewing lectures 7-8, then talk about homework 4.

## Review

In this section, we review capacitors, the method of images, and separation of variables in the context of electrostatics as discussed in lectures 7-8.

**Capacitors (problem 2&3)** Consider two conductors with opposite charges in otherwise empty space.

- By Coulomb's law (following the notations in lecture 7 p. 4)

$$V_Q(\vec{x}) = \sum_{n=1}^2 \int_{C_n} \frac{da'}{4\pi\epsilon_0} \frac{\sigma_n(\vec{x}')}{|\vec{x} - \vec{x}'|} \quad (1)$$

with the conductors charged  $Q$  and  $-Q$ , the electric potential is *uniquely* determined.

- Outside the conductors we have Laplace's equation  $\nabla^2 V_Q = 0$ .
- The capacitance defined as

$$C \equiv \frac{Q}{V_Q(1) - V_Q(2)} \quad (2)$$

is *independent* of the charge  $Q$  (lecture 7 pp. 4-5).

- Here  $V_Q(1)$  denotes the potential of the positively charged conductor and  $V_Q(2)$  denotes that of the negatively charged conductor.
- Typical recipe of computing capacitance of a two-conductor system:
  - Assume one conductor carries positive charge  $Q > 0$  and the other one carries  $-Q$ .
  - Express the potential difference  $V_Q(1) - V_Q(2)$  in terms of  $Q$  and geometric factors (usually using Coulomb's law).
  - Use eq. (2) to compute the capacitance.

- Capacitor property

$$Q = CV \quad (3)$$

with  $V$  the potential difference  $V \equiv V_Q(1) - V_Q(2)$ .

- Capacitor energy

$$W = \frac{Q^2}{2C} = \frac{1}{2}CV^2 \quad (4)$$

- Electric force

$$\vec{F}_{\text{elec}} = -\vec{\nabla}_{x_1} W(x_1). \quad (5)$$

**Uniqueness theorem** The uniqueness theorem is the key to the method of image and separation of variable. The statement of the uniqueness theorem is that (lecture 6 p.7):

The solution to Laplace's equation  $\nabla^2 V = 0$  in some volume  $\mathcal{V}$  is uniquely determined if  $V$  is specified on the boundary surface  $\partial\mathcal{V}$ .

**The method of images (problem 3-5)** Consider an electrostatic problem in a region  $\mathcal{V}$  (physical region) with boundary conditions specified on the boundary surface  $\partial\mathcal{V}$ .

- The method of images is to *guess* a solution to Laplace's equation  $\nabla^2 V = 0$  by placing an image charge in unphysical region (outside  $\mathcal{V}$ ), such that the given boundary condition on  $\partial\mathcal{V}$  is satisfied.
- The electric potential  $V$  in the physical region  $\mathcal{V}$  is then given by the superposition of the electric potential generated by physical charges and image charges.
- The uniqueness theorem guarantees that the potential  $V$  in the physical region  $\mathcal{V}$  generated by the method of images is the *only* solution.
- Important examples:
  - Lecture 8 pp. 6-8 (grounded conducting plane), where  $q' = -q$  and  $d' = d$ .
  - Lecture 8 pp. 9-10 (grounded conducting sphere), where  $q' = -(R/a)q$  and  $b = R^2/a$ .

**Separation of variable (problem 6&7)** The context for separation of variable is solving Laplace's equation  $\nabla^2 V(x_1, x_2, x_3) = 0$  for a given boundary condition and symmetries.

- First, take the ansatz

$$V(x_1, x_2, x_3) = A(x_1)B(x_2)C(x_3). \quad (6)$$

The intuition here is that the potential  $V$  varies independently for each coordinate.

- For example, in Cartesian coordinates, with the separation of variable  $V(x, y, z) = X(x)Y(y)Z(z)$ , Laplace's equation becomes (Griffith, example 3.5)

$$0 = \nabla^2 V \Rightarrow \frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = 0. \quad (7)$$

- It follows that

$$\frac{1}{X} \frac{d^2 X}{dx^2} = C_1, \quad \frac{1}{Y} \frac{d^2 Y}{dy^2} = C_2, \quad \frac{1}{Z} \frac{d^2 Z}{dz^2} = C_3, \quad \text{with } C_1 + C_2 + C_3 = 0. \quad (8)$$

- Solve  $X$ ,  $Y$ , and  $Z$  with given boundary conditions.

- Finding Fourier coefficients on the interval  $[0, a]$ .

- Consider the Fourier series (e.g. lecture 8 pp. 13-18)

$$f(y) = \sum_{n'=1}^{\infty} C_{n'} f_{n'}(y), \quad f_{n'}(y) = \sin\left(\frac{n'\pi y}{a}\right). \quad (9)$$

- Note that the sinusoidal functions possess the orthogonality property

$$\int_0^a dy f_n(y) f_{n'}(y) = \int_0^a dy \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n'\pi y}{a}\right) = \frac{a}{2} \delta_{nn'}. \quad (10)$$

- We solve the coefficients  $C_n$  via the projection

$$\int_0^a dy f(y) f_n(y) = \int_0^a dy \left[ \sum_{n'=1}^{\infty} C_{n'} f_{n'}(y) \right] f_n(y) = \sum_{n'=1}^{\infty} C_{n'} \frac{a}{2} \delta_{nn'} = \frac{a}{2} C_n, \quad (11)$$

which yields

$$C_n = \frac{2}{a} \int_0^a dy f(y) f_n(y). \quad (12)$$


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## Homework 4

The problems in homework 4 can be classified as:

- Ideal conductor (prob. 1)
- Capacitor (prob. 2, 3)
- The method of images (prob. 4, 5)
- Separation of variable (prob. 6, 7)

In the handout I will give you hint on each problem, but for the sake of time, during the discussion sessions we will only talk about problems 2, 5, and 7 (or maybe less).

### Ideal conductor

Recall that an ideal conductor is an equipotential. The subquestions in problem 1 can be solved using this equipotential property and electrostatic equations. (Gauss's law, Poisson's equation etc.)

**Problem 1a** Show that any net charge on a conductor must lie on its surface.

- Conductor  $\rightarrow$  equipotential  $\rightarrow \vec{E} = 0$  inside the conductor.
- Charge density  $\rho$  is given by Gauss's law  $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$ .
  - Given that  $\vec{E} = 0$  inside the conductor, what can you say about  $\rho$  inside?

**Problem 1b** Show that a closed hollow conductor shields its interior from fields due to exterior charge, but does not shield the exterior from the charge on the interior. Call the hollow region  $R$ .

- Shielding interior from fields due to exterior charge means that if you place any charge on the exterior, the electric field in the hollow is always zero.
  - On the interior: Laplace's equation  $\nabla^2 V = 0$  and boundary condition  $V = \text{constant}$  on  $\partial R$ . (the boundary of the hollow)
  - The solutions on the interior is *uniquely* determined due to the uniqueness theorem.
    - \* What is the solution? Can you derive  $\vec{E} = 0$  in  $R$  then?
- Outside fields are not shielded from the charges from the inside: See lecture 6 p. 9.

**Problem 1c** Use the pill-box type of argument to show the boundary condition of  $\vec{E}$ :

- Normal component

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \Rightarrow E^\perp(+)-E^\perp(-) = \frac{\sigma}{\epsilon_0}. \quad (13)$$

- Parallel component

$$\vec{\nabla} \times \vec{E} = 0 \Rightarrow E^\parallel(+)-E^\parallel(-) = 0. \quad (14)$$

These should suffice to solve the problem. (You DO need to prove them to get full credits.)

## Capacitor

**Problem 2a** This problem is about calculating the capacitance of a capacitor composed of two concentric cylinders.

- Recipe for calculating capacitance of a two conductor capacitor:
  - Put fictitious charges  $Q$  and  $-Q$  on each conductor.
  - Calculate the potential difference  $V_Q(1) - V_Q(2)$  from one conductor to the other.
  - The capacitance is then given by  $C = Q/(V_Q(1) - V_Q(2))$ .
- For this problem, we follow the above recipe.
  - Put a fictitious charge  $Q$  on the inner cylinder and  $-Q$  on the outer cylinder.
  - To calculate the potential difference from the inner cylinder to the outer cylinder, we work out the electric field  $\vec{E}$  between them first.
    - \* Because of the cylindrical symmetry, we choose cylindrical coordinates  $(s, \phi, z)$ .
    - \* Due to the symmetry, we find  $E_\phi = E_z = 0$  in the overlapping region, as  $b - a \ll l \ll$  length of cylinders.
    - \* Compute  $E_s$  using Gauss's law. (take a cylindrical Gaussian surface, also check the hint for problem 4a).
    - \* Integrate  $E_s$  along the radial direction to get  $V_Q(1) - V_Q(2)$ .
    - \* Capacitance  $C = Q/(V_Q(1) - V_Q(2))$ .

**Problem 3** This problem is to calculate the force by considering the change in energy of the capacitor system.

- You may use the capacitor energy and electric force formulae

$$W = \frac{Q^2}{2C}, \quad \vec{F}_{\text{elec}} = -\vec{\nabla}_{\vec{x}_1} W(\vec{x}_1). \quad (15)$$

- Your final answer can contain  $f$  and its derivatives, and a fictitious charge  $Q$ .

## The method of images

**Problem 4a** The first task here is to guess the image in the unphysical region  $\{z < 0\}$ , with the boundary condition  $V|_{z=0} = 0$  satisfied.

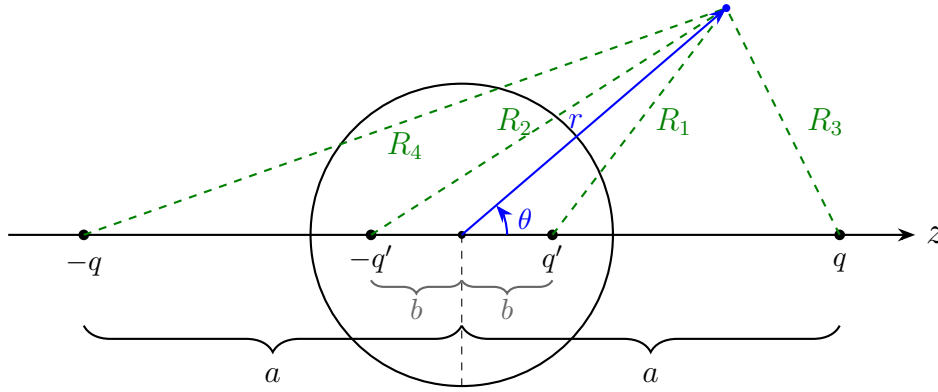
- Consider an infinitesimal segment of the line charge. What is its image?
- Then take a guess, what is the image of the line?
- Verify that your guess can satisfy the boundary condition  $V|_{z=0} = 0$  using the  $V_{\text{single}}$  formula.

**Problem 5** Let us first elucidate the setup of the problem.

- Place a spherical conductor of radius  $R$  in an otherwise uniform electric field  $\vec{E} = E_0 \hat{z}$ .
  - This corresponds to the boundary condition  $V = -E_0 z + C$  as  $|z| \rightarrow \infty$ .
- The induced charge disturbs the electric field around the conductor,
  - but far away from the conductor, i.e.,  $|z| \rightarrow \infty$ , we still have  $V \approx -E_0 z + C$ .
- The problem is asking you to find the “disturbed” potential  $V(r, \theta)$  outside the conductor using the method of images.

The aside for this problem is telling you how to generate the boundary condition  $V \approx -E_0 z + C$  as  $|z| \rightarrow \infty$  using two point charges. With this aside, what you need to do in this problem are

1. Figure out the images of the two charges  $q$  and  $-q$ , i.e., express  $q'$  and  $b$  in terms of  $q$  and  $a$ .
2. Find the potential  $V(r, \theta)$  outside the conductor using Coulomb’s law.
3. Find  $V(r, \theta)$  in the limit  $a \rightarrow \infty$  with  $q/a^2 = -2\pi\epsilon_0 E_0$ .



Here I give you a hint on the last two steps. From geometry, we see that

$$R_1 = \sqrt{(r \cos \theta - b)^2 + r^2 \sin^2 \theta}, \quad R_2 = \sqrt{(r \cos \theta + b)^2 + r^2 \sin^2 \theta} \quad (16)$$

$$R_3 = \sqrt{(r \cos \theta - a)^2 + r^2 \sin^2 \theta}, \quad R_4 = \sqrt{(r \cos \theta + a)^2 + r^2 \sin^2 \theta}. \quad (17)$$

As  $a \rightarrow \infty$ , we note  $r/a \rightarrow 0$ . Then, for example,

$$R_3 = a \sqrt{\left(1 - \frac{r \cos \theta}{a}\right)^2 + \frac{r^2 \sin^2 \theta}{a^2}} \approx a \sqrt{1 - \frac{2r \cos \theta}{a}}. \quad (18)$$

View  $r \cos \theta/a$  as the small quantity  $g$ . With  $\sqrt{1+g} = 1 + g/2 + O(g^2)$  we can approximate  $R_3$  as

$$R_3 = a \left(1 - \frac{r \cos \theta}{a} + O\left(\frac{r^2}{a^2}\right)\right) \approx a \left(1 - \frac{r \cos \theta}{a}\right). \quad (19)$$

The potential outside is given by Coulomb’s law

$$V(r, \theta) = \frac{q'}{4\pi\epsilon_0 R_1} + \frac{-q'}{4\pi\epsilon_0 R_2} + \frac{q}{4\pi\epsilon_0 R_3} + \frac{-q}{4\pi\epsilon_0 R_4}. \quad (20)$$

The last step is to find the approximate expression for  $V$  in the limit  $a \rightarrow \infty$ .

## Separation of variable

**Problem 6** This is a variation of the example on lecture 8 pp. 13-17.

- Start with the expression on lecture 8 p. 16 (you may cite this in your HW)

$$V = \sum_{n=1}^{\infty} \left[ \frac{2}{a} \int_0^a d\tilde{y} \sin\left(\frac{n\pi\tilde{y}}{a}\right) V_0(\tilde{y}) \right] e^{-\frac{n\pi}{a}x} \sin\left(\frac{n\pi}{a}y\right). \quad (21)$$

- Give that

$$V_0(y) = \begin{cases} V_0 & \text{if } 0 \leq y < a/2 \\ -V_0 & \text{if } a/2 \leq y < a \end{cases}, \quad (22)$$

and the property

$$\int_0^a dy \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n'\pi y}{a}\right) = \frac{a}{2} \delta_{nn'}, \quad (23)$$

compute

$$\int_0^a d\tilde{y} \sin\left(\frac{n\pi\tilde{y}}{a}\right) V_0(\tilde{y}). \quad (24)$$

**Problem 7** Start with separation of variables in Cartesian coordinates  $V(x, y, z) = X(x)Y(y)Z(z)$ , after which Laplace's equation becomes

$$\frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = 0, \quad (25)$$

giving

$$\frac{1}{X} \frac{d^2 X}{dx^2} = C_1, \quad \frac{1}{Y} \frac{d^2 Y}{dy^2} = C_2, \quad \frac{1}{Z} \frac{d^2 Z}{dz^2} = C_3, \quad \text{with } C_1 + C_2 + C_3 = 0. \quad (26)$$

- What are the values of  $C_1$ ,  $C_2$  and  $C_3$ ? Depends on the boundary condition.
- Note the  $X(x)$  and  $Y(y)$  solutions vanish on the box walls, they should be sinusoidal.
  - Hence, we choose negative  $C_1$  and  $C_2$  to generate sinusoidal  $X(x)$  and  $Y(y)$ .
  - Then  $C_3 = -C_1 - C_2 > 0$ , the solution  $Z(z)$  is a superposition of  $e^{\pm\sqrt{C_3}z}$ .
- To conclude, the separated differential equations read

$$X'' = -c_x X, \quad Y'' = -c_y Y, \quad Z'' = (c_x + c_y)Z \quad (27)$$

where  $c_x \equiv -C_1$  and  $c_y \equiv -C_2$ .

- Next, use the boundary conditions on the  $x$  and  $y$  directions to determine the values of  $c_x$  and  $c_y$ , then obtain an expression for  $Z(z)$  from  $Z'' = (c_x + c_y)Z$ .
  - Use the boundary condition on the  $z$  direction to fix all the unknown coefficients.
- Your final answer should be an infinite sum over two integer indices. You can also check Griffith's example 3.5 for inspiration.