PHY322 S26
Discussion 5
TA: Junqi Zhang

In this discussion, we will start with reviewing lectures 9-10, then talk about homework 5.

Review

In this section, we review spherical coordinate Laplace solutions and multipole expansion in the context of electrostatics, as discussed in lectures 9-10.

Spherical coordinate Laplace solutions (with azimuthal symmetry) The spherical coordinate Laplace solutions $V(r,\theta,\phi)$ are obtained by separation of variables as follows.

- Start with writing down Laplace's equation in spherical coordinates

$$0 = \nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} \tag{1}$$

- Make a separation of variables $V(r,\theta,\phi) = R(r)\Theta(\theta)$ with azimuthal symmetry.
  - Afterwards, eq. (1) becomes

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = 0\,. \tag{2}$$

  - The two terms in the above equation are constants, say

$$0 = \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - l(l+1) \tag{3}$$

$$0 = \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + l(l+1)\,. \tag{4}$$

  - General solutions for R and Θ are

$$R(r) = Ar^l + \frac{B}{r^{l+1}}, \quad \Theta(\theta) = c_l P_l(\cos\theta) \tag{5}$$

    with $P_l(\cos\theta)$ the Legendre polynomial.

- Therefore, the spherical coordinate Laplace solutions with azimuthal symmetry reads

$$\boxed{V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)}\,. \tag{6}$$

- Useful properties for the Legendre polynomial:
  - Rodrigues formula (Griffiths' eq. (3.62))

$$P_l(x) \equiv \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l\,. \tag{7}$$

  - First few terms:$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = (3x^2-1)/2$, and $P_3(x) = (5x^3-3x)/2$.
  - Orthogonality:

$$\int_{-1}^{1} d\cos\theta P_l(\cos\theta)P_{l'}(\cos\theta) = \frac{2}{2l+1}\delta_{ll'}\,. \tag{8}$$

Multipole expansion Multipole expansion is a systematic approximation approach to see how the field of a local charge distribution $\rho(\vec{r'})$ behaves at a far distance r with $|\vec{r}| \gg |\vec{r'}|$. I will list the key results below, and for derivations of them please see lecture 10.

- Start with Coulomb's law for a localized charge distribution

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d\tau' \frac{\rho(\vec{r'})}{|\vec{r} - \vec{r'}|} \,. \tag{9}$$

- Note for $|\vec{r}| \gg |\vec{r'}|$ we have the approximation (lecture 10 p. 5)

$$\frac{1}{|\vec{r} - \vec{r'}|} = \sum_{l=0}^{\infty} \frac{r'^l}{r^{l+1}} P_l(\cos\alpha) \tag{10}$$

  where $r \equiv |\vec{r}|,\ r' \equiv |\vec{r'}|$ and α is the angle between $\vec{r}$ and $\vec{r'}$.

- As a result, the potential at $\vec{r}$ can be expanded as

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \int d\tau' \rho(\vec{r'}) r'^l P_l(\cos\alpha) \tag{11}$$

$$= \frac{1}{4\pi\epsilon_0} \Big[\underbrace{\frac{1}{r} \int d\tau' \rho(\vec{r'})}_{\text{monopole}} + \underbrace{\frac{1}{r^2} \int d\tau' r' \cos\alpha \rho(\vec{r'})}_{\text{dipole}} + \underbrace{\frac{1}{r^3} \int d\tau' r'^2 \left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right) \rho(\vec{r'})}_{\text{quadrupole}} + \cdots \Big] . \tag{12}$$

- We define the l-th multipole as

$$q_l \equiv \int d\tau' \rho(\vec{r'}) r'^l P_l(\cos\alpha) \,. \tag{13}$$

- For dipole, we define the dipole moment as a vector

$$\vec{P} \equiv \int d\tau' \vec{r'} \rho(\vec{r'}) \,. \tag{14}$$

  - Dipole potential

$$V_{\text{dipole}}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{P} \cdot \hat{r}}{r^2} \,. \tag{15}$$

  - Dipole electric field

$$\vec{E}_{\text{dipole}}(r, \theta) = \frac{P}{4\pi\epsilon_0 r^3} \left(2\cos\theta\, \hat{r} + \sin\theta\, \hat{\theta} \right) \,. \tag{16}$$

  - Consider an external field $\vec{E}$ acting on $\rho(\vec{x})$ centered about $\vec{x} = 0$, and suppose that $\vec{E}$ is approximately uniform around the charges (lecture 10 pp. 9-10).
    * Total force acting on the charge by $\vec{E}$ is

$$\vec{F}_t = Q\vec{E}(0) + (\vec{P} \cdot \vec{\nabla})\vec{E}|_{\vec{x}=0} \,. \tag{17}$$

    * Torque acting on the charge by $\vec{E}$ is

$$\vec{N} = \vec{P} \times \vec{E}(0) \,. \tag{18}$$

Homework 5

The first two problems of homework 5 are about spherical coordinate Laplace solutions and the other five are about multipole expansions.

Spherical coordinate Laplace solutions

Problem 1a The potential on a spherical boundary of radius R is $V_0 = k\cos 3\theta$ with k constant. Find the potential everywhere.

- As a first step, we write down the general spherical coordinate Laplacian solutions

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta), \quad r \le R \tag{19}$$

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta), \quad r > R\,. \tag{20}$$

- Use the boundary condition $V(r=R,\theta) = k\cos 3\theta$ to fix the A_l and B_l coefficients.
  - For example, approaching $r = R$ from the interior we have

$$V(R,\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = k\cos 3\theta\,. \tag{21}$$

  - How to figure out A_l? Well, in principle you could use the orthogonality of Legendre polynomials (8) and make a projection onto $P_m(\cos\theta)$ via integration.
  - Acute folks might notice that instead of performing integrations of $\cos 3\theta$ times Legendre polynomials, we could decompose $\cos 3\theta$ over Legendre polynomials first, i.e., writing down the decomposition

$$\cos 3\theta = a_0 + a_1 P_1(\cos\theta) + a_2 P_2(\cos\theta) + a_3 P_3(\cos\theta) \tag{22}$$

    with the constants a_1, a_2 and a_3 determined by a matching.
    - Hint: the following trigonometrical identities might be useful

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{23}$$

$$\cos 2\theta = 2\cos^2\theta - 1\,. \tag{24}$$

  - With decomposition (22), eq. (21) then tells you that

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = ka_0 + ka_1 P_1(\cos\theta) + ka_2 P_2(\cos\theta) + ka_3 P_3(\cos\theta)\,. \tag{25}$$

  - Since Legendre polynomials are orthogonal,

$$A_0 = ka_0, \quad A_1 R = ka_1, \quad A_2 R^2 = ka_2, \quad \ldots \tag{26}$$

  - How to figure out the B_l coefficients? Hint: $V(r,\theta)$ is continuous at $r = R$.

Problem 1b Find the charge density on the spherical boundary. This is a from-field-to-charge type of question. You may use (lecture 9 p. 13)

$$\left[\frac{\partial}{\partial r}V(+) - \frac{\partial}{\partial r}V(-)\right]_{r=R} = -\frac{\sigma}{\epsilon_0}\,. \tag{27}$$

Multipole expansion

Problem 3a In lecture 10 p. 5 you have learned the multipole expansion formula

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{f_l(\hat{r})}{r^{l+1}}, \quad f_l(\hat{r}) \equiv \int d\tau' \rho(\vec{r}')r'^l P_l(\cos\alpha), \quad \cos\alpha \equiv \hat{r}\cdot\hat{r}'\,. \tag{28}$$

We know the solution to Laplace's equation with azimuthal symmetry is

$$V(\vec{r}) = \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta)\,. \tag{29}$$

Find B_l in terms of $f_l(\hat{z})$.

- We match eqs. (28) and (29) on the z-axis

$$V(r\hat{z}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{f_l(\hat{z})}{r^{l+1}} = \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta)|_{\theta=0}\,. \tag{30}$$

- What are B_l coefficients in terms of $f_l(\hat{z})$ then?

Problem 4 Consider a linear charge density distribution

$$\rho = \begin{cases}\Omega\left(3\left[\frac{z}{L}\right]^2 - 1\right)\delta(x)\delta(y) & z\in[-L,L]\\ 0 & z\notin[-L,L]\end{cases} \tag{31}$$

in Cartesian coordinates where Ω is a constant. Find the leading term in the multipole expansion of the potential.

- This problem is to exercise the multipole expansion formula (slightly different notations)

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{q_l}{r^{l+1}}, \quad q_l \equiv \int d\tau'\rho(\vec{r}')r'^l P_l(\cos\alpha), \quad \cos\alpha\equiv\hat{r}\cdot\hat{r}'\,. \tag{32}$$

- We compute q_l:

$$\begin{aligned}
q_l &= \int dx'dy'dz'\rho(\vec{r'})r'^l P_l(\hat{r}\cdot\hat{r}') && (33)\\
&= \int_{-L}^{L} dx'dy'dz'\Omega\left(3\left[\frac{z'}{L}\right]^2 - 1\right)\delta(x')\delta(y')r'^l P_l(\hat{r}\cdot\hat{r}') && (34)\\
&= \int_0^L dz'\Omega\left(3\left[\frac{z'}{L}\right]^2 - 1\right)z'^l P_l(\hat{r}\cdot\hat{z}) + \int_{-L}^{0} dz'\Omega\left(3\left[\frac{z'}{L}\right]^2 - 1\right)z'^l P_l(-\hat{r}\cdot\hat{z}) && (35)\\
&= \int_0^L dz'\Omega\left(3\left[\frac{z'}{L}\right]^2 - 1\right)z'^l P_l(\cos\theta) + \int_{-L}^{0} dz'\Omega\left(3\left[\frac{z'}{L}\right]^2 - 1\right)z'^l P_l(-\cos\theta) && (36)
\end{aligned}$$

where in the second last line we have used $\hat{r}'|_{x=0,\,y=0,\,z>0} = \hat{z}$ and $\hat{r}'|_{x=0,\,y=0,\,z<0} = -\hat{z}$.

- To further simplify eq. (36), use $P_l(-x) = (-1)^l P_l(x)$ (derivable from eq. (7)).
- Compute the first few q_l terms. The first nonzero one is what we call "the leading term".

Problem 5a The potential on the positive z-axis of a uniformly charged disk of radius R sitting at $z = 0$ is

$$V(r, \theta = 0) = \frac{\sigma}{2\epsilon_0}\left(\sqrt{r^2 + R^2} - r\right) \tag{37}$$

where σ is the charge density. For $r > R$, find the potential for all θ to quadrupole moment accuracy.

- To find the potential for all θ, you may first calculate it on the z-axis, then use the conclusion in problem 3a to generalize the result to all θ.
- On the z-axis, we may expand the potential at large r to $O(1/r^3)$ power (quadrupole accuracy)

$$V(r, \theta = 0) = V(r\hat{z}) = \frac{\sigma r}{2\epsilon_0}\left(\sqrt{1 + \left(\frac{R}{r}\right)^2} - 1\right) \approx \frac{a_0}{r} + \frac{a_1}{r^3} + \dots \tag{38}$$

  - Hint: To find a_0 and a_1, you may use the expansion for $g \ll 1$:

$$\sqrt{1+g} \approx 1 + \frac{1}{2}g - \frac{1}{8}g^2 \,. \tag{39}$$

- Next, for $r > R$, matching $V(r\hat{z}) \approx a_0/r + a_1/r^3$ to

$$V(r, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \tag{40}$$

  we find that for all θ (following the idea in problem 3)

$$V(r, \theta) \approx \frac{a_0}{r} P_0(\cos\theta) + \frac{a_1}{r^3} P_2(\cos\theta) \,.$$

Problem 6a Consider a two dipole model (originated from water molecules model) where one dipole $\vec{p}_1 = (\hat{x} + \hat{y})p/\sqrt{2}$ sits at the origin and the other dipole $\vec{p}_2 = p\hat{r}$ is at $\vec{r} = d_1\hat{x} + d_2\hat{y} + d_3\hat{z}$. Find the dipole torque on $\vec{p}_2$ due to $\vec{p}_1$ dipole electric field.

- We first calculate the $\vec{p}_1$ dipole electric potential with

$$V_{1,\,\mathrm{dip}} = \frac{\hat{r} \cdot \vec{p}_1}{4\pi\epsilon_0 r^2} = \frac{p}{4\pi\epsilon_0 r^2} \frac{\hat{r} \cdot (\hat{x} + \hat{y})}{\sqrt{2}} = \dots \tag{41}$$

- The $\vec{p}_1$ dipole electric field is then given by $\vec{E}_{1,\,\mathrm{dip}} = -\vec{\nabla} V_{1,\,\mathrm{dip}}$, which can be computed in spherical coordinates.
- The torque is thus

$$\vec{N} = \vec{p}_2 \times \vec{E}_{1,\,\mathrm{dip}} \tag{42}$$