

The main goal of this discussion is to help you prepare for midterm 1.

Review

In the first third of the semester, we have learned the first three chapters in Griffiths' electrodynamics: vector analysis, electrostatics and potentials. Here I present a more detailed list.

Vector analysis

This chapter is about mathematical terminology that will be used in analyzing physical problems.

- The Del operator ∇ , gradient, divergence, curl and Laplacian in various coordinates (Cartesian, cylindrical and spherical coordinates). Vector identities.

- Stokes theorem

$$\int_V d\tau (\vec{\nabla} \cdot \vec{A}) = \oint_{\partial V} d\vec{a} \cdot \vec{A}, \quad \int_S d\vec{a} \cdot (\vec{\nabla} \times \vec{A}) = \oint_{\partial S} d\vec{l} \cdot \vec{A} \quad (1)$$

- Delta function

1. Normalization $\int_{-\infty}^{\infty} dx \delta(x - x_0) = 1$.
2. Pick out points $\int_{-\infty}^{\infty} dx \delta(x - x_0) f(x) = f(x_0)$.
3. A useful theorem

$$\int_D d\tau h(\vec{x}) \delta^{(3)}(\vec{f}(\vec{x})) = \sum_n \frac{h(\vec{x}_{*n})}{\left| \det \left(\frac{\partial f_i}{\partial x_j} \right) \right|_{\vec{x}=\vec{x}_{*n}}} \quad (2)$$

where \vec{x}_{*n} are the zeros of \vec{f} contained in D .

4. A useful identity

$$\nabla^2 \left(\frac{1}{r} \right) = -\nabla \cdot \left(\frac{\hat{r}}{r^2} \right) = -4\pi \delta^{(3)}(\vec{r}) \quad (3)$$

- Vector field decomposition

– For a vector function \vec{F} ,

$$\vec{\nabla} \times \vec{F} = 0 \quad \rightarrow \quad \vec{F} = \vec{\nabla} S, \quad (4)$$

$$\vec{\nabla} \cdot \vec{F} = 0 \quad \rightarrow \quad \vec{F} = \vec{\nabla} \times \vec{A}. \quad (5)$$

Electrostatics

Our exposition of electrostatics started with Coulomb's law, which tells you how charges give rise to electric fields \vec{E} , then we upgraded it to a more compact form, often referred to as Gauss's law. With the vector field decomposition theorems, we introduced a scalar potential V such that $\vec{E} = -\vec{\nabla}V$, which simplifies the task of finding the vector field \vec{E} to finding a scalar field V . In the homework problems, we emphasized how to utilize symmetry in practice to simplify the work.

With these preparations, we introduced conductors defined as objects that are equipotentials, then capacitance. Also, we considered how to find electric force through energy. Among these topics, the uniqueness theorem and boundary conditions play a critical role.

Fields and potentials

- Coulomb's law

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}, \quad \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\vec{r}')(\hat{r} - \hat{r}')}{|\vec{r} - \vec{r}'|^2}. \quad (6)$$

- Gauss's law

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \quad \oint d\vec{a} \cdot \vec{E} = \frac{Q_{\text{enc}}}{\epsilon_0}. \quad (7)$$

- From Coulomb's law we find $\vec{\nabla} \times \vec{E} = 0$. According to the vector decomposition theorem, there exists a scalar potential V such that $\vec{E} = -\vec{\nabla}V$.

- Poisson's equation is the obtained from Gauss's law

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}. \quad (8)$$

- Electric field boundary conditions

- Normal component

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \Rightarrow E^\perp(+)-E^\perp(-) = \frac{\sigma}{\epsilon_0}. \quad (9)$$

- Parallel component

$$\vec{\nabla} \times \vec{E} = 0 \Rightarrow E^\parallel(+)-E^\parallel(-) = 0. \quad (10)$$

- The uniqueness theorem: The solution to Laplace's equation $\nabla^2 V = 0$ in some volume \mathcal{V} is uniquely determined if V is specified on the boundary surface $\partial\mathcal{V}$.

Conductors

- An ideal conductor is an object of equipotential ($V = \text{constant}$) that can carry charge.
- Consider a two-conductor system. The capacitance defined as

$$C \equiv \frac{Q}{V_Q(1) - V_Q(2)} \quad (11)$$

is *independent* of the charge Q .

- Here $V_Q(1)$ denotes the potential of the positively charged conductor and $V_Q(2)$ denotes that of the negatively charged conductor.

- Capacitor property

$$Q = CV \quad (12)$$

with V the potential difference $V \equiv V_Q(1) - V_Q(2)$.

- Capacitor energy

$$W = \frac{Q^2}{2C} = \frac{1}{2}CV^2 \quad (13)$$

- Electric force

$$\vec{F}_{\text{elec}} = -\vec{\nabla}_{x_1} W(\vec{x}_1). \quad (14)$$

Potentials

This chapter comprises two topics:

1. Solving the Laplace equation $\nabla^2 V = 0$ with various techniques.
2. Multipole expansion.

Let's enumerate the important things here.

Techniques for solving the Laplace equation

- The method of images. Consider an electrostatic problem in a region \mathcal{V} (physical region) with boundary conditions specified on the boundary surface $\partial\mathcal{V}$.
 - The method of images is to *guess* a solution to Laplace's equation $\nabla^2 V = 0$ by placing an image charge in unphysical region (outside \mathcal{V}), such that the given boundary condition on $\partial\mathcal{V}$ is satisfied.
 - The electric potential V in the physical region \mathcal{V} is then given by the superposition of the electric potential generated by physical charges and image charges.
 - The uniqueness theorem guarantees that the potential V in the physical region \mathcal{V} generated by the method of images is the *only* solution.
 - Important examples: Lecture 8 pp. 6-8 (grounded conducting plane), where $q' = -q$ and $d' = d$. Lecture 8 pp. 9-10 (grounded conducting sphere), where $q' = -(R/a)q$ and $b = R^2/a$.
- Separation of variables. The context for separation of variables is solving Laplace's equation $\nabla^2 V(x_1, x_2, x_3) = 0$ for a given boundary condition and symmetries. We usually use the ansatz $V(x_1, x_2, x_3) = A(x_1)B(x_2)C(x_3)$.
 - Cartesian coordinates. See Griffiths' 3.3.1.
 - Spherical coordinates with azimuthal symmetry (i.e., $V(\vec{r})$ is independent of ϕ).
 - * General solutions to Laplace's equation can be written as

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta). \quad (15)$$

Multipole expansion Multipole expansion is a systematic approximation approach to see how a local charge distribution $\rho(\vec{r}')$ behaves at a far distance r with $|\vec{r}| \gg |\vec{r}'|$.

- Start with Coulomb's law for a localized charge distribution

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d\tau' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}. \quad (16)$$

- Note for $|\vec{r}| \gg |\vec{r}'|$ we have the approximation (lecture 10 p.5)

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{l=0}^{\infty} \frac{r'^l}{r^{l+1}} P_l(\cos \alpha) \quad (17)$$

where $r \equiv |\vec{r}|$, $r' \equiv |\vec{r}'|$ and α is the angle between \vec{r} and \vec{r}' .

- As a result, the potential at \vec{r} can be expanded as

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \int d\tau' \rho(\vec{r}') r'^l P_l(\cos \alpha) \quad (18)$$

$$= \frac{1}{4\pi\epsilon_0} \left[\underbrace{\frac{1}{r} \int d\tau' \rho(\vec{r}')}_{\text{monopole}} + \underbrace{\frac{1}{r^2} \int d\tau' r' \cos \alpha \rho(\vec{r}')}_{\text{dipole}} + \underbrace{\frac{1}{r^3} \int d\tau' r'^2 \left(\frac{3}{2} \cos^2 \alpha - \frac{1}{2} \right) \rho(\vec{r}')}_{\text{quadrupole}} + \dots \right]. \quad (19)$$

- We define the l -th multipole as

$$q_l \equiv \int d\tau' \rho(\vec{r}') r'^l P_l(\cos \alpha). \quad (20)$$

- For dipole, we define the dipole moment as a vector

$$\vec{P} \equiv \int d\tau' \vec{r}' \rho(\vec{r}'). \quad (21)$$

- Dipole potential

$$V_{\text{dipole}}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{P} \cdot \hat{r}}{r^2}. \quad (22)$$

- Dipole electric field

$$\vec{E}_{\text{dipole}}(r, \theta) = \frac{P}{4\pi\epsilon_0 r^3} \left(2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right). \quad (23)$$

Midterm 1

Since you have access to practice midterm 1 solutions, I will not talk about them here. Here are two preparation tips:

- Prioritize reviewing your homework. Ideally, you should be able to solve every problem independently, without relying on any references.
- Review your lecture notes and work through the examples again from scratch.

Good luck :)