PHY322 S26
Discussion 7
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Homework 6 covers a wide range of topics, including dipole force and torque, dielectrics and the Lorentz force. As usual, we will start with reviewing lectures 11-13, then discuss homework 6 problems.

Review

In this section, we will focus on reviewing dielectrics and will only present formula lists for the dipole, the Lorentz force and the current topics.

Dipole

Here we provide a formula list for dipoles.

- Dipole moment
$$\vec{P} \equiv \int d\tau' \vec{r'} \rho(\vec{r'}) \,. \tag{1}$$
- Dipole potential
$$V_{\text{dipole}}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{P} \cdot \hat{r}}{r^2} \,. \tag{2}$$
- Dipole electric field
$$\vec{E}_{\text{dipole}}(r, \theta) = \frac{P}{4\pi\epsilon_0 r^3} \left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right) \,. \tag{3}$$
- Consider an external field $\vec{E}$ acting on $\rho(\vec{x})$ centered about $\vec{x} = 0$, and suppose that $\vec{E}$ is approximately uniform around the charges (lecture 10 pp. 9-10).
  - Total force acting on the charge by $\vec{E}$ is
$$\vec{F}_t = Q\vec{E}(0) + (\vec{P} \cdot \vec{\nabla})\vec{E}|_{\vec{x}=0} \,. \tag{4}$$
  - Torque acting on the charge by $\vec{E}$ is
$$\vec{N} = \vec{P} \times \vec{E}(0) \,. \tag{5}$$

Dielectrics (electric fields in matter)

One motivation for investigating electrostatic properties of dielectrics is to understand how electric fields behave in matter. What we have learnt so far are the equations governing static electric fields and charges in vacuum or conductors. In vacuum, Coulomb's law dictates electrostatic phenomena whereas in conductors the equipotential property suffices to explain many scenarios.

Electrostatics in matter (dielectrics), however, can be rather sophisticated due to complicated microscopic structures. This means it is difficult to determine electric potentials or fields at atomic scales. Nevertheless, it is still possible to establish electrostatic equations in matter at a macroscopic scale, by utilizing the dipole approximation (see Griffiths' Sec. 4.1 for details).

In this section, we will start with articulating our dielectric model, then derive Gauss's law and boundary conditions in the presence of dielectrics. We will review the concepts including electric displacement, bound charge and free charge, and permittivity along the way.

- Dielectric model. Approximate the charges in matter as dipoles with dipole moment $\vec{P}_i$, and define a macroscopic polarization density $\vec{P}$ as [1]

$$\vec{P} \equiv \text{dipole moment per volume} = \frac{\sum_i \vec{P}_i}{\mathcal{V}} \,. \tag{6}$$

  Here $\vec{P}$ is intuitively the 'dipole moment density'.

  - Superposing all the dipole potentials gives (cf. eq. (2))

$$V_{\text{pol}}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} d\tau' \frac{\vec{P}(\vec{r'}) \cdot \left(\hat{r} - \hat{r'}\right)}{|\vec{r} - \vec{r'}|^2} \,. \tag{7}$$

  - Noting $\vec{\nabla}_{\vec{r'}} \left(1/|\vec{r} - \vec{r'}|\right) = (\hat{r} - \hat{r'})/|\vec{r} - \vec{r'}|^2$, we rewrite $V_{\text{pol}}(\vec{r})$ and integrate by parts

$$V_{\text{pol}}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} d\tau' \vec{P}(\vec{r'}) \cdot \vec{\nabla}_{\vec{r'}} \left(\frac{1}{|\vec{r} - \vec{r'}|} \right) = \cdots \tag{8}$$

$$= \frac{1}{4\pi\epsilon_0} \oint_{\partial\mathcal{V}} da' \frac{\sigma_b}{|\vec{r} - \vec{r'}|} + \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} d\tau' \frac{\rho_b(\vec{r'})}{|\vec{r} - \vec{r'}|} \,, \tag{9}$$

    where we have defined $\sigma_b \equiv \vec{P} \cdot \hat{n}$ as the effective surface charge and $\rho_b \equiv -\vec{\nabla}_{\vec{r'}} \cdot \vec{P}(\vec{r'})$ as the effective volume charge.

- Gauss's law in dielectrics. In dielectrics, the electric potential can be calculated from two perspectives.

  - On one hand, given the total charge density ρ, we have Coulomb's law

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} d\tau' \frac{\rho(\vec{r'})}{|\vec{r} - \vec{r'}|} \,. \tag{10}$$

  - On the other hand, the potential in dielectrics is attributed to the dipole pairs and everything else

$$V(\vec{r}) = V_{\text{pol}}(\vec{r}) + V_{\text{other}}(\vec{r}) \,. \tag{11}$$

  - We call the charges other than the dipole pairs the free charge ρ_f, and write

$$V_{\text{other}}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} d\tau' \frac{\rho_f(\vec{r'})}{|\vec{r} - \vec{r'}|} \,. \tag{12}$$

  - Comparing the above results, we have the classification of charges

$$\rho = \rho_b + \rho_f \,. \tag{13}$$

[1] Note this is an abuse of the notation $\vec{P}$. It means either the dipole moment or the polarization density, depending upon the context.

– Gauss's law implies

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} = \frac{\rho_b + \rho_f}{\epsilon_0} \,. \tag{14}$$

* Plugging the effective volume charge definition $\rho_b \equiv -\vec{\nabla} \cdot \vec{P}$ into the above gives

$$\vec{\nabla} \cdot \vec{E} = \frac{-\vec{\nabla} \cdot \vec{P} + \rho_f}{\epsilon_0} \quad \rightarrow \quad \vec{\nabla} \cdot \left(\epsilon_0 \vec{E} + \vec{P}\right) = \rho_f \tag{15}$$

* Introducing the electric displacement $\vec{D} \equiv \epsilon_0 \vec{E} + \vec{P}$, we obtain Gauss's law in matter

$$\boxed{\vec{\nabla} \cdot \vec{D} = \rho_f} \,. \tag{16}$$

* Integrated Gauss's law in matter

$$\oint d\vec{a} \cdot \vec{D} = Q_{f\,\text{enclosed}} \,. \tag{17}$$

- Linear dielectrics are materials in which the following relation is satisfied

$$\vec{P} = \epsilon_0 \chi_e \vec{E} \tag{18}$$

where χ_e is the electric susceptibility.

– Defining permittivity $\epsilon \equiv \epsilon_0(1 + \chi_e)$, with which

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon \vec{E} \,. \tag{19}$$

– Dielectric constant $\epsilon_r \equiv \epsilon/\epsilon_0 = 1 + \chi_e$.

- Boundary conditions in dielectrics. Using a "pill-box" type of argument, we find

– Normal component

$$\vec{\nabla} \cdot \vec{D} = \rho_f \quad \Rightarrow \quad D^{\perp}(+) - D^{\perp}(-) = \sigma_f \,. \tag{20}$$

– Parallel component

$$\vec{\nabla} \times \vec{E} = 0 \quad \Rightarrow \quad E^{\parallel}(+) - E^{\parallel}(-) = 0 \,. \tag{21}$$

Lorentz force and currents

- The Lorentz force formula reads $\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$.
- Current density $\vec{J} \equiv \rho \vec{v}$, and surface current density $\vec{K} = \sigma \vec{v}$.

Homework 6

The first two problems of homework 6 are about dipole force and torque, problems 3-5 are dielectric exercises, problem 6 is solving the charge motion due to Lorentz force and problem 7 is about currents.

Dipole

Problem 1 Consider a two dipole model (originated from water molecules model) where one dipole $\vec{p}_1 = (\hat{x} + \hat{y})p/\sqrt{2}$ sits at the origin and the other dipole $\vec{p}_2 = p\hat{r}$ is at $\vec{r} = d_1\hat{x} + d_2\hat{y} + d_3\hat{z}$. Find the dipole torque on $\vec{p}_2$ due to $\vec{p}_1$ dipole electric field.

- We first calculate the $\vec{p}_1$ dipole electric potential with

$$V_{1,\,\mathrm{dip}} = \frac{\hat{r}\cdot\vec{p}_1}{4\pi\epsilon_0 r^2} = \frac{p}{4\pi\epsilon_0 r^2}\frac{\hat{r}\cdot(\hat{x}+\hat{y})}{\sqrt{2}} = \ldots \tag{22}$$

- The $\vec{p}_1$ dipole electric field is then given by $\vec{E}_{1,\,\mathrm{dip}} = -\vec{\nabla}V_{1,\,\mathrm{dip}}$, which can be computed in spherical coordinates.

- The torque is thus

$$\vec{N} = \vec{p}_2 \times \vec{E}_{1,\,\mathrm{dip}}\,.$$

Dielectrics

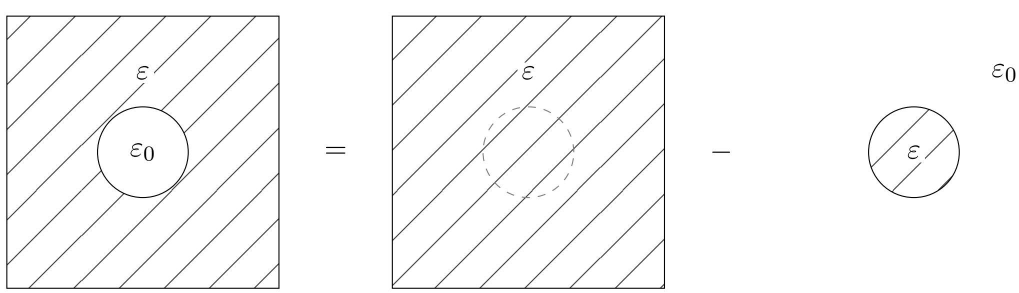
Problem 3 Suppose the electrostatic field inside a large piece of dielectric having polarization $\vec{P}$ is a constant $\vec{E}_0$.

3a A small spherical cavity of radius R is made inside this material by hollowing this shape out. Find the electric field at the center of the cavity.

- Recall that in lecture 11 pp. 4-5 we have solved that for a uniformly polarized sphere with polarization density $\vec{P}$, the electric field inside is

$$\vec{E}_{\mathrm{ball}} = -\frac{\vec{P}}{3\epsilon_0}\,. \tag{23}$$

- Note the superposition



$$\vec{E}_1 \qquad = \qquad \vec{E}_0 \qquad - \qquad \vec{E}_{\mathrm{ball}}$$

where $\vec{E}_1$ denotes the left over electric field at the center.

3b What is the $\vec{D}$ at the center? Hint: use the relation (19).

3c and 3d Redo 3a and 3b for a cylindrical cavity. Hint: follow the logic in 3a and 3b but utilize the approximation conditions.

Problem 4 An electric dipole moment $p\hat{z}$ is located at the center of a spherical uniform dielectric shell with inner radius A, outer radius B and permittivity ϵ. Find the potential everywhere. You are given the general solution in the form

$$V_1 = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta), \quad 0 \le r \le A \tag{24}$$

$$V_2 = \sum_{l=0}^{\infty} \left(C_l r^l + \frac{D_l}{r^{l+1}} \right) P_l(\cos\theta), \quad A \le r \le B \tag{25}$$

$$V_3 = \sum_{l=0}^{\infty} \frac{E_l}{r^{l+1}} P_l(\cos\theta), \quad r \ge B\,. \tag{26}$$

4a As $r \to 0$, we know $V_1(r,\theta) \to V_{\text{dip}} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$. From this, determine B_l.

- From $V_1 = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta), \quad 0 \le r \le A$, we see that as $r \to 0$,

$$V_1 \to \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}, \quad 0 \le r \le A\,. \tag{27}$$

- Matching coefficients gives you B_l.
- After finding B_l you should obtain

$$V_1 = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} + \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta), \quad 0 \le r \le A\,. \tag{28}$$

4b Write 3 algebraic equations corresponding to potential continuity at $r = A$.

- Note the potential continuity at $r = A$ reads $V_1(r = A, \theta) = V_2(r = A, \theta)$, giving

$$\frac{p\cos\theta}{4\pi\epsilon_0 A^2} + \sum_{l=0}^{\infty} A_l A^l P_l(\cos\theta) = \sum_{l=0}^{\infty} \left(C_l A^l + \frac{D_l}{A^{l+1}} \right) P_l(\cos\theta)\,. \tag{29}$$

- Split the above relation into $l = 0, 1$, and $l \ge 2$ equations.

4d Write 3 equations for $D^\perp$ continuity at $r = A$.

- Since there are no free charges on the boundary, $D^\perp$ is continuous.
- Note $D^\perp = \epsilon E^\perp$ (here ϵ depends on r). How to express $E^\perp$ in terms of potential derivatives?

Problem 5 Consider a cylindrical capacitor made up of an inner metal cylinder of radius a and an outer metal cylinder of radius b. The longitudinal length of the metal cylinder is $L > l$. A toilet-paper-roll shaped linear dielectric material (which is an insulator) of permittivity ϵ partially fills the capacitor gap, covering a length l in the longitudinal direction.

5a Find the capacitance. Neglect edge effects as usual.

- Recall the capacitance definition $C = Q/(V_Q(1) - V_Q(2))$.
- Assign fictitious charges Q to the inner conductor and $-Q$ to the outer conductor.
  - To compute the potential difference $V = V_Q(1) - V_Q(2)$, suppose $Q = q_1 + q_2$ with q_1 the charge in the region without the dielectric, and q_2 the charge in the region with the dielectric.
  - By cylindrical symmetry, the electric field only has radial component $\vec{E} = E_s \hat{s}$.
  - At the boundary of the dielectric $E^{\parallel}$ is continuous, meaning that E_s is the same across the two regions.
  - In the region without dielectric, Gauss's law gives $E_s = \frac{q_1}{2\pi\epsilon_0 s(L-l)}$.
  - In the region with dielectric, Gauss's law gives $D_s = \frac{q_2}{2\pi s l}$. With $D_s = \epsilon E_s$, you can solve E_s in the region with dielectric.
  - Find the potential difference V in both regions. Express q_1 and q_2 in terms of V.
  - Compute $C = \frac{Q}{V} = \frac{q_1+q_2}{V}$. You should find V canceled in the final result.

Lorentz force and currents

Problem 6 Review example 5.2 in lecture 13. The techniques used in the example and problem 6 are isomorphic.

Problem 7 Use the current density definitions $\vec{K} = \sigma\vec{v}$ (surface current density) and $\vec{J} = \rho\vec{v}$ (volume current density).