

In this discussion we will start by reviewing the basic ingredients for magnetostatics covered in lectures 13-15, then talk about homework 7.

## Review

Our investigation of magnetostatics begins with the Lorentz force formula and current density. Analogous to Coulomb's law in electrostatics, we introduced the Biot-Savart law which tells you how steady currents dictate magnetic fields. Then we moved to Ampère's law and introduced the vector potential, based on which a set of boundary conditions for the magnetic field vector potential was derived.

### Lorentz force and currents

Here I present a formula list for the Lorentz force and current densities.

- The Lorentz force for a point charge  $q$  is

$$\vec{F} = q \left( \vec{E} + \vec{v} \times \vec{B} \right). \quad (1)$$

- Magnetic force on a wire (lecture 13 p. 12) and a volume current (lecture 14 p. 3)

$$\vec{F}_{\text{mag}}^{\text{wire}} = \int \left( d\vec{l} \times \vec{B} \right) I, \quad \vec{F}_{\text{mag}}^{\text{volume current}} = \int d^3x \vec{J} \times \vec{B}. \quad (2)$$

- Current density  $\vec{J} \equiv \rho \vec{v}$ , and surface current density  $\vec{K} = \sigma \vec{v}$ .

### The Biot-Savart law

Coulomb's law tells you that for a point charge  $q$ , the electric field is given by  $\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\hat{r}-\hat{r}'}{|\vec{r}-\vec{r}'|^2}$ .

- Similarly, in the context of magnetostatics, the Biot-Savart law is the counterpart of Coulomb's law, stating that for a wire with current  $I$ , the magnetic field is (given by experiments)

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}' \times (\hat{r} - \hat{r}')}{|\vec{r} - \vec{r}'|^2}, \quad (3)$$

where  $\mu_0 = 4\pi \times 10^{-7} \text{ N/A}^2$  in SI units.

- For a volume current density  $\vec{J}(\vec{r}')$  or surface current density  $\vec{K}(\vec{r}')$ , the magnetic field is given by the integral

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int d\tau' \frac{\vec{J}(\vec{r}') \times (\hat{r} - \hat{r}')}{|\vec{r} - \vec{r}'|^2}, \quad \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int da' \frac{\vec{K}(\vec{r}') \times (\hat{r} - \hat{r}')}{|\vec{r} - \vec{r}'|^2}, \quad (4)$$

respectively.

- Review the examples in lecture 14 on how to compute  $\vec{B}$  for various configurations.

## Ampère's law, magnetic vector potential, and boundary conditions

Recall that in electrostatics, we obtain Gauss's law  $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$  by taking the divergence of Coulomb's law. By taking the curl of Coulomb's law we get  $\vec{\nabla} \times \vec{E} = 0$  and that implies the existence of electric potential. Here we repeat this process to find  $\vec{\nabla} \cdot \vec{B}$  and  $\vec{\nabla} \times \vec{B}$  using the Biot-Savart law (4).

- Taking the divergence of eq. (4) yields

$$\vec{\nabla} \cdot \vec{B} = 0, \quad (5)$$

which implies the existence of a magnetic vector potential  $\vec{A}$  such that  $\vec{B} = \vec{\nabla} \times \vec{A}$ .

- Since  $\vec{\nabla} \times (\vec{\nabla}\Lambda) = 0$  is true for any smooth scalar field  $\Lambda$ , we uncover a gauge symmetry

$$\vec{A} \rightarrow \vec{A} + \vec{\nabla}\Lambda \quad (6)$$

under which the magnetic field  $\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times (\vec{A} + \vec{\nabla}\Lambda)$  is invariant.

- Utilizing the freedom in choosing  $\vec{\nabla}\Lambda$ , we may always set  $\vec{\nabla} \cdot \vec{A} = 0$  (Coulomb gauge).

- Taking the curl of eq. (4) yields

$$\boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}}, \quad (7)$$

which is Ampère's law.

- Analogous to the integrated Gauss's law  $\oint d\vec{a} \cdot \vec{E} = Q_{\text{enc}}/\epsilon_0$  in electrostatics, taking a surface integral of eq. (7) yields

$$\oint d\vec{l} \cdot \vec{B} = \mu_0 I_{\text{enclosed}}, \quad (8)$$

where we have used Stokes' theorem.

- Plugging the magnetic vector potential expression  $\vec{B} = \vec{\nabla} \times \vec{A}$  into Ampère's law we find

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}, \quad (9)$$

which is akin to Poisson's equation  $\nabla^2 V = -\rho/\epsilon_0$  in electrostatics. We have seen that the solution to Poisson's equation is exactly the Coulomb law expression  $V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d\tau' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}$ . Now the question is, can we find an analogous solution to eq. (9)?

- The answer is yes. Since in Cartesian coordinates each component of eq. (9) has exactly the same form as Poisson's equation in electrostatics, we find

$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int d\tau' \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|}}. \quad (10)$$

- For line and surface currents

$$\vec{A}_{\text{line}}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}'}{|\vec{r} - \vec{r}'|}, \quad \vec{A}_{\text{surface}}(\vec{r}) = \frac{\mu_0}{4\pi} \int da' \frac{\vec{K}(\vec{r}')}{|\vec{r} - \vec{r}'|}. \quad (11)$$

- Boundary conditions in magnetostatics

- Normal component

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \Rightarrow \quad B_{\perp}(+) - B_{\perp}(-) = 0. \quad (12)$$

- Parallel component

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad \Rightarrow \quad B_{\parallel}^{(\perp K)}(+) - B_{\parallel}^{(\perp K)}(-) = \mu_0 K. \quad (13)$$

- Combining the above two gives

$$\vec{B}(+) - \vec{B}(-) = \mu_0 (\vec{K} \times \hat{n}). \quad (14)$$

- For the magnetic vector potential the above boundary conditions translate to

$$\vec{A}(+) - \vec{A}(-) = 0 \quad (15)$$

$$(\hat{n} \cdot \vec{\nabla})\vec{A}(+) - (\hat{n} \cdot \vec{\nabla})\vec{A}(-) = -\mu_0 \vec{K}. \quad (16)$$

## Homework 7

All problems in homework 7 are about calculating the magnetic field or the magnetic vector potential, using either the Biot-Savart law or Ampère's law. Note the last two problems emphasize the magnetic vector potential properties.

### The Biot-Savart law

Problems 1, 3, and 4 may be solved using the Biot-Savart law. A typical strategy in solving them is partitioning a complicated current configuration into simple components, calculate the magnetic field or magnetic vector potential of each component, then superpose them.

**Problem 1** Find the magnetic field at point  $C$  at the center of a square current loop of side length  $L$  with current  $I$ .

- We work in Cartesian coordinates centered at  $C$  with  $x$  and  $y$  axes aligned with the bottom and right side currents, as shown in the figure in the problem set.
- Consider the contribution from the right side  $\{x = L/2, y \in [L/2, -L/2]\}$ .
  - Use the Biot-Savart law

$$\vec{B}_R(\vec{r}) = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}' \times (\hat{r} - \hat{r}')}{|\vec{r} - \vec{r}'|^2} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}' \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \quad (17)$$

where we have rewritten using  $\hat{r} - \hat{r}' = (\vec{r} - \vec{r}')/|\vec{r} - \vec{r}'|$ .

– Here  $d\vec{l}' = dy'\hat{y}$ ,  $\vec{r}' = 0$ , and  $\vec{r} = \frac{L}{2}\hat{x} + y'\hat{y}$ , giving

$$\vec{B}_R(\vec{r} = 0) = \frac{\mu_0 I}{4\pi} \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dy'\hat{y} \times (0 - (\frac{L}{2}\hat{x} + y'\hat{y}))}{|0 - (\frac{L}{2}\hat{x} + y'\hat{y})|^3} = \dots \quad (18)$$

– Hint: You will encounter the integral

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dy'}{[(\frac{L}{2})^2 + y'^2]^{3/2}}. \quad (19)$$

Consider the change of variable  $y' = \frac{L}{2} \tan \theta$ .

- Superpose the magnetic contributions from the right, left, top and bottom sides.

**Problem 3** Consider two identical circular loops of radii  $R$  carrying currents  $I$ . They center about the  $z$  axis and are placed at  $z = \pm R/2$ .

**3a** Find the magnetic field at any point along the  $z$  axis.

- In lecture 14 we have worked out the magnetic field at a distance  $z$  above a single circular loop of radius  $R$  carrying current  $I$ , which reads (you will use this in problem 4 as well)

$$\vec{B}_{\text{loop}}(z) = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}} \hat{z}. \quad (20)$$

- Superpose the magnetic field contribution from the two loops

$$\vec{B}(z) = \vec{B}_{\text{loop}}\left(\frac{R}{2} + z\right) + \vec{B}_{\text{loop}}\left(\frac{R}{2} - z\right). \quad (21)$$

**3d** Suppose you are designing an experiment with this coil configuration in which you want the field along the axis to be constant to 10% accuracy. How big is the length interval centered about the origin where this is achieved?

- We need to estimate  $\vec{B}(z)$  around  $z = 0$  using Taylor expansion, which gives

$$\vec{B}(z) = \vec{B}(0) + z \left[ \frac{\partial \vec{B}}{\partial z} \right]_{z=0} + \frac{1}{2!} z^2 \left[ \frac{\partial^2 \vec{B}}{\partial z^2} \right]_{z=0} + \frac{1}{3!} z^3 \left[ \frac{\partial^3 \vec{B}}{\partial z^3} \right]_{z=0} + \frac{1}{4!} z^4 \left[ \frac{\partial^4 \vec{B}}{\partial z^4} \right]_{z=0} + \dots \quad (22)$$

- Find the first nonzero contribution in the above Taylor series.
- The math is straightforward but the calculation is cumbersome. Feel free to use Mathematica or SymPy to double check your result.

## Magnetic vector potential

Problem 6 is about calculating the magnetic vector potential for a circular loop. Problem 7 focuses on the gauge symmetry aspect of the magnetic vector potential.

**Problem 6** Consider a loop of radius  $a$  centered about the origin and lying in the  $xy$  plane, carrying current  $C$ .

**6a** Evaluate the vector potential  $\vec{A}$  at the location  $(0, 0, z)$  (here the coordinates are Cartesian).

- We use eq. (11) (note here the current is denoted by  $C$ )

$$\vec{A}(\vec{r}) = \frac{\mu_0 C}{4\pi} \oint \frac{d\vec{l}'}{|\vec{r} - \vec{r}'|}. \quad (23)$$

- Here  $d\vec{l}' = d[a(\cos \phi' \hat{x} + \sin \phi' \hat{y})] = (\dots)d\phi'$  and from geometry  $|\vec{r} - \vec{r}'| = \sqrt{a^2 + z^2}$ . Then

$$\vec{A}(\vec{r}) = \frac{\mu_0 C}{4\pi} \oint \frac{(\dots)d\phi'}{\sqrt{a^2 + z^2}}. \quad (24)$$

**6b** Evaluate the vector potential  $\vec{A}$  at the location  $(s, 0, 0)$  (here the coordinates are Cartesian) in terms of  $f(w) \equiv \int_0^{2\pi} \frac{d\phi' \cos \phi'}{\sqrt{1-w \cos \phi'}}$ .

- Again, use eq. (11)

$$\vec{A}(\vec{r}) = \frac{\mu_0 C}{4\pi} \oint \frac{d\vec{l}'}{|\vec{r} - \vec{r}'|}. \quad (25)$$

- Note we still have  $d\vec{l}' = d[a(\cos \phi' \hat{x} + \sin \phi' \hat{y})] = (\dots)d\phi'$ , while  $|\vec{r} - \vec{r}'| = \sqrt{a^2 + s^2 - 2as \cos \phi'}$ . Then you may compute the  $y$  component of the vector potential using

$$A_y = \hat{y} \cdot \vec{A}(\vec{r}) = \frac{\mu_0 C}{4\pi} \oint \frac{\hat{y} \cdot (\dots)d\phi'}{\sqrt{a^2 + s^2 - 2as \cos \phi'}} = \dots. \quad (26)$$

**Problem 7c** Find a gauge function  $\Lambda$  satisfying

$$\vec{\underline{A}} = \vec{A} + \vec{\nabla}\Lambda \quad (27)$$

(giving  $\vec{B} = \vec{\nabla} \times \vec{A} = -2B_0 \hat{z}$ ) where  $\vec{\underline{A}}$  has no  $\hat{x}$  or  $\hat{z}$  components,  $\vec{\underline{A}}$  vanishes at  $(x, y, z) = (L, 0, 0)$ , and  $\vec{A}$  is given by  $\vec{A} = B_0 y \hat{x} - B_0 x \hat{y}$ . Restrict  $\Lambda$  to be functions which for any fixed  $x$  value is linear in  $y$  and vanishes at  $y = 0$ .

- The objective of this problem is to gauge transform the vector potential  $\vec{A}$  to another vector potential  $\vec{\underline{A}}$  with certain properties to be satisfied.
- The desired properties for  $\vec{\underline{A}}$  are  $\underline{A}_x = \underline{A}_z = 0$  and  $\underline{A}|_{(x,y,z)=(L,0,0)} = 0$ .
  - The  $x$  and  $z$  components of Eq. (27) then imply

$$0 = B_0 y + \partial_x \Lambda, \quad 0 = 0 + \partial_z \Lambda. \quad (28)$$

- The restriction on  $\Lambda$  suffices to solve the above differential equations with all integral constants fixed.