PHY322 S26
Discussion 9
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In this discussion we will first review the magnetic dipole moment and magnetic fields in matter as covered in lectures 16-17, then talk about Homework 8.

Review

The theory of magnetic fields in matter closely parallels that of electric fields. We begin with a multipole expansion of the magnetic vector potential to define the microscopic magnetic dipole. To investigate macroscopic fields, we then average these microscopic dipoles. Just as polarization is defined as the electric dipole moment per unit volume, its magnetic counterpart, magnetization, is the magnetic dipole moment per unit volume. This framework allows us to translate the concepts of bound charge and electric displacement into bound current and the auxiliary magnetic field, respectively. A set of magnetic boundary conditions is also derived.

Magnetic dipole moment

In this section we review the definition of magnetic dipole moment for a closed loop with a current I. Remember the electric and magnetic dipoles are tools for investigating how the fields behave at a distance very far away from the source.

- Recall that the magnetic vector potential $\vec{A}$ is given by

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{l'}}{|\vec{r} - \vec{r'}|}\,. \tag{1}$$

- Recall the expansion for $|\vec{r'}| \ll |\vec{r}|$ introduced for the multipole expansion of electric potential

$$\frac{1}{|\vec{r} - \vec{r'}|} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos\alpha) \tag{2}$$

  where α is the angle between $\vec{r}$ and $\vec{r'}$.

- It follows that

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \oint d\vec{l'} (r')^n P_n(\cos\alpha) \tag{3}$$

$$= \frac{\mu_0 I}{4\pi} \left[\underbrace{\cancelto{0}{\frac{1}{r}\oint d\vec{l'}}}_{\text{monopole}} + \underbrace{\frac{1}{r^2}\oint d\vec{l'} r' \cos\alpha}_{\text{dipole}} + \underbrace{\frac{1}{r^3}\oint d\vec{l'} (r')^2 \left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)}_{\text{quadrupole}} + \cdots \right]. \tag{4}$$

  - The magnetic monopole term vanishes.

  - The *magnetic dipole* term, after some rewriting as shown in lecture 16, is

$$\boxed{\vec{A}_{\text{dip}}(\vec{r}) = \frac{\mu_0}{4\pi r^2}\vec{m} \times \hat{r}, \quad \vec{m} \equiv I \int_{\Sigma} d\vec{a'}}\,, \tag{5}$$

    where Σ denotes the surface enclosed by the current loop.
  - The magnetic field induced by the dipole is

$$\vec{B}_{\text{dip}}(\vec{r}) = \vec{\nabla} \times \vec{A}_{\text{dip}} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right] . \tag{6}$$

  - In spherical coordinates

$$\vec{A}_{\text{dip}}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi}, \quad \vec{B}_{\text{dip}}(\vec{r}) = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right) . \tag{7}$$

- Torques and forces on a magnetic dipole
  - Torque $\vec{N} = \vec{m} \times \vec{B}$.
  - Force $\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})$.

Magnetic fields in matter

Here we review how magnetic fields behave in matter.

- Introduce the magnetization (intuitively the magnetic moment density)

$$\vec{M}(\vec{r}) \equiv \text{magnetic moment per volume}\,. \tag{8}$$

- The magnetic vector potential induced by the magnetization is

$$\begin{aligned} \vec{A}(\vec{r}) &= \frac{\mu_0}{4\pi}\int d\tau' \frac{\vec{M}(\vec{r'}) \times (\hat{r} - \hat{r'})}{|\vec{r} - \vec{r'}|^2} & (9)\\ &= \ldots \text{ (lecture 17, pp. 2-3)} & (10)\\ &= \frac{\mu_0}{4\pi}\left(\int_{\mathcal{V}} d\tau' \frac{\vec{\nabla}' \times \vec{M}(\vec{r'})}{|\vec{r} - \vec{r'}|} + \oint_{\partial\mathcal{V}} da' \frac{\vec{M}(\vec{r'}) \times \hat{n'}}{|\vec{r} - \vec{r'}|}\right) . & (11) \end{aligned}$$

- Introducing a bound current density $\vec{J_b}(\vec{r'}) \equiv \vec{\nabla}' \times \vec{M}(\vec{r'})$ and a bound surface current $\vec{K_b}(\vec{r'}) \equiv \vec{M}(\vec{r'}) \times \hat{n'}$, we get

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\left(\int_{\mathcal{V}} d\tau' \frac{\vec{J_b}(\vec{r'})}{|\vec{r} - \vec{r'}|} + \oint_{\partial\mathcal{V}} da' \frac{\vec{K_b}(\vec{r'})}{|\vec{r} - \vec{r'}|}\right) . \tag{12}$$

- Now we introduce an auxiliary field as we did in introducing the electric displacement.
  - Start with Ampère's law $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$.
  - Define the free current density $\vec{J_f}$ as $\vec{J_f} \equiv \vec{J} - \vec{J_b}$. This gives a current density decomposition

$$\vec{J} = \vec{J_f} + \vec{J_b}\,. \tag{13}$$

- Plugging $\vec{J}_b = \vec{\nabla} \times \vec{M}$ into Ampère's law gives

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_f + \mu_0 \vec{\nabla} \times \vec{M} \quad \to \quad \vec{\nabla} \times \left(\frac{\vec{B}}{\mu_0} - \vec{M} \right) = \vec{J}_f \,. \tag{14}$$

- Define the auxiliary field as

$$\vec{H} \equiv \frac{\vec{B}}{\mu_0} - \vec{M} \tag{15}$$

  We then obtain Ampère's law in terms of H

$$\vec{\nabla} \times \vec{H} = \vec{J}_f \,. \tag{16}$$

- Note $\vec{\nabla} \cdot \vec{B} = 0$ is still true.

- For linear media, we have a linear response relation

$$\vec{M} = \chi_m \vec{H} \tag{17}$$

  where χ_m is called the magnetic susceptibility.

  - The magnetic field

$$\vec{B} = \mu_0 \left(\vec{H} + \vec{M} \right) = \mu_0 \left(\vec{H} + \chi_m \vec{H} \right) \equiv \mu \vec{H}, \tag{18}$$

    where $\mu \equiv \mu_0 (1 + \chi_m)$ is the permeability.

- Boundary conditions
  - Normal component

$$\vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M} \quad \Rightarrow \quad \vec{H}_\perp(+) - \vec{H}_\perp(-) = -\left(\vec{M}_\perp(+) - \vec{M}_\perp(-) \right) , \tag{19}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \Rightarrow \quad B_\perp(+) - B_\perp(-) = 0 \,. \tag{20}$$

  - Parallel component

$$\vec{\nabla} \times \vec{H} = \vec{J}_f \quad \Rightarrow \quad \vec{H}_\parallel(+) - \vec{H}_\parallel(-) = \vec{K}_f \times \hat{n} \,, \tag{21}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad \Rightarrow \quad \vec{B}(+) - \vec{B}(-) = \mu_0 \left(\vec{K} \times \hat{n} \right) \,. \tag{22}$$

Homework 8

Homework 8 problems can be grouped as follows

- Magnetic dipoles: problems 2 and 4.
- Magnetic fields in matter: problems 5-7.
- Miscellaneous: problems 1 and 3.

We will focus on the first two groups.

Magnetic dipole

In the magnetic dipole exercises, you will compute magnetic dipole moments, magnetic fields induced by dipoles, and torques.

Problem 2 Consider two planar current loops with one tilted at an angle θ. Details are given in the problem set.

2a Approximating each as an ideal magnetic dipole, with the left one located at the origin and the right one located at $\vec{x} = H\hat{y}$, what is the torque on the right loop due to the magnetic field of the left loop?

- First, calculate the magnetic dipole moments $\vec{m}_{L/R}$ of the left/right loop using

$$\vec{m} = I \int_{\Sigma} d\vec{a}' \, . \tag{23}$$

  - What is the direction of $\vec{m}_R$? How to decompose $\hat{e}$ into the Cartesian basis $\{\hat{x}, \hat{y}, \hat{z}\}$?

- The magnetic field induced by the left loop is

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\vec{m}_L \cdot \hat{r})\hat{r} - \vec{m}_L\right] \, , \tag{24}$$

  where $\vec{r}$ is the vector pointing from the location of the left loop to the right loop.

  - What is $\vec{r}$ in this setup?

- The torque on the right loop is then

$$\vec{N} = \vec{m}_R \times \vec{B} \, . \tag{25}$$

2c Make an argument why $\theta = -\pi/2$ is an unstable equilibrium even though the torque does vanish there.

- A system is in an unstable equilibrium if an infinitesimally small perturbation generates a net force or torque that accelerates it away from that initial state.
- What is the torque for the perturbed angle $\theta = -\pi/2 + \epsilon$? Is the torque pushing the right loop away or dragging it back from $\theta = -\pi/2$?

Problem 4 Again we have a dipole^2 system. Calculations for this problem are straightforward applications of the following formulae

- Magnetic vector potential and field for a magnetic dipole

$$\vec{A}_{\text{dip}}(\vec{r}) = \frac{\mu_0}{4\pi r^2}\vec{m} \times \hat{r}, \quad \vec{B}_{\text{dip}}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m}\right] \, . \tag{26}$$

- Force on magnetic dipole $\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$.

Magnetic fields in matter

For the sake of time I will focus on problem 5 in the discussion, but as usual, I will provide hints on problems 6 and 7.

Problem 5 A spherical shell of inner radius a and outer radius b is constructed of uniformly magnetized material with magnetization $\vec{M}$.

5a Find the equivalent volume and surface current distributions.

- Recall that for magnetic fields in matter, the bound volume current density is $\vec{J}_b = \vec{\nabla} \times \vec{M}$ and the bound surface current density is $\vec{K}_b = \vec{M} \times \vec{n}$.
  - In spherical coordinates, how to express $\hat{n}_{\text{outer}}$ and $\hat{n}_{\text{inner}}$ in terms of the basis $\{\hat{r}, \hat{\theta}, \hat{\phi}\}$?

5b Find the magnetic field $\vec{B}$ everywhere.

- Think of the spherical shell as a superposition of a uniformly magnetized ball of radius a with $\vec{M} = -M\hat{z}$ and another uniformly magnetized ball of radius b with $\vec{M} = M\hat{z}$.
- To find the magnetic field of each magnetized ball, recall example 5.11 and lecture 17 pp. 13-14.
  - Example 5.11: A spherical shell of radius R, carrying uniform surface charge σ, is set spinning at angular velocity $\vec{\omega}$. The magnetic field is

$$\vec{B}(\vec{r}) = \begin{cases} \frac{2}{3}\mu_0 \sigma R \vec{\omega} & r < R \\ \frac{2\mu_0 \omega R^4 \sigma}{3r^3}\left(\cos\theta \hat{r} + \frac{1}{2}\sin\theta\hat{\theta}\right) & r \geq R \end{cases}. \tag{27}$$

  - Using the argument in lecture 17, replacing $\sigma\omega$ with M/R gives the magnetic field of a uniformly magnetized ball

$$\vec{B}(\vec{r}) = \begin{cases} \frac{2}{3}\mu_0 \vec{M} & r < R \\ \frac{\mu_0 |\vec{m}_{\text{total}}|}{2\pi r^3}\left(\cos\theta \hat{r} + \frac{1}{2}\sin\theta\hat{\theta}\right) & r \geq R \end{cases}, \quad |\vec{m}_{\text{total}}| \equiv \frac{4\pi R^3}{3} M\,. \tag{28}$$

- Superpose the two magnetized balls with opposite magnetizations to get $\vec{B}$ everywhere.

5c Find the auxiliary field $\vec{H}$ everywhere.

- Use $\vec{H} = \vec{B}/\mu_0 - \vec{M}$ and the results from 5b.

Problem 6 Consider a sphere of magnetic material with magnetization $\vec{M} = \hat{z}Mz/R$ centered about the origin having radius R. Compute the magnetic field at the point $(0, 0, R)$ [this point is in Cartesian coordinates]. Express your answer as

$$\vec{B} = \frac{\mu_0 M}{2}\hat{z}\int_0^\pi d\theta f(\theta)\,. \tag{29}$$

- As in problem 5, the first step is to find the bound currents using

$$\vec{J}_b = \vec{\nabla} \times \vec{M}, \quad \vec{K}_b = \vec{M} \times \vec{n}\,. \tag{30}$$

  - You will find $\vec{J}_b = 0$ and $\vec{K}_b \propto \cos\theta \sin\theta \hat{\phi}$.

- This is equivalent to a superposition of rings, with each ring carrying $dI = K_b R d\theta$.
- The magnetic field at a point above the center of a ring of radius R carrying current I is

$$\vec{B}_{\text{loop}}(z) = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}} \hat{z}\,. \tag{31}$$

- Superpose all the rings to get the magnetic field at $(0, 0, z = R)$ by integrating

$$d\vec{B} = \frac{\mu_0 dI}{2} \frac{R^2}{(R^2 + z^2)^{3/2}} \hat{z}\,. \tag{32}$$

Problem 7 Hint: Analyze the symmetry and integrate Ampère's law for the auxiliary field $\vec{\nabla} \times \vec{H} = \vec{J}_f$.

Miscellaneous

It is hard to say which topic problems 1 and 3 belong to. Nevertheless, I will provide a hint for problem 3.

Problem 3a You are asked to solve the magnetic vector potential $\vec{A}$ with

$$-\nabla^2 \vec{A} = \mu_0 \vec{J} \tag{33}$$

using Gauss's law analog technique.

- To find the analog, note for each component $\nabla^2 A^i = \vec{\nabla} \cdot \vec{\nabla} A^i$, which means

$$\vec{\nabla} \cdot \vec{\nabla} A^i = -\mu_0 J^i\,. \tag{34}$$

- Compare this with Gauss's law

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}\,. \tag{35}$$

  - We may view $\vec{\nabla} A^i$ as the analog of $\vec{E}$ (up to a constant).